Exploring SO(3) logarithmic map: degeneracies and derivatives

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Summary

1. Explored SO(3) log map: Rodrigues, S-matrix, Quaternion
2. Derived Jacobians of extended log maps for optimization algorithms and proved its correctness

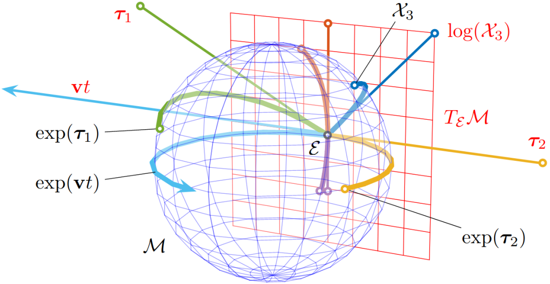

Figure: Representation of the relation between the Lie group and the Lie algebra. [Solà et al., 2020]

Motivation

SLAM pose-graph optimization, rotational part of residual [Youyang et al., 2020]:

$$r_{ij}(\mathbf{R}_i, \mathbf{R}_j) = \log(\Delta R_{ij}^{-1} \mathbf{R}_i^{-1} \mathbf{R}_j)^{\vee} \tag{1}$$

In optimization frameworks like Ceres [Agarwal et al.,]:

- Rotation is modeled as a manifold by associating the corresponding tangent space via `LocalParameterization` interface, defining $\boxplus$ operator and its Jacobian at zero:

$$\frac{df(R)}{dR} \left. \frac{d\left(R \boxplus \mathbf{x}\right)}{d\mathbf{x}} \right|_{\mathbf{x}=\mathbf{0}} \tag{2}$$

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Goals

1. Numerically stable evaluation of log map at all points, including degenerate case of rotation angle θ close or equal to π.
2. Define the Jacobian of extended log map and prove its correctness.

$$J_{\overline{\log}}(R) \equiv \frac{d\overline{\log}(R)}{dR} \in \mathbb{R}^{3\times 9}$$

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SO(3) exponential map

$$\exp : \mathfrak{so}(3) \mapsto \mathbf{SO}(3)$$
$$\mathbb{R}^3 \ni \boldsymbol{\omega} \mapsto \mathbf{R}_{3\times 3}$$

Rodrigues' formula [Rodrigues, 1840]:

$$\exp(\boldsymbol{\omega}) \equiv \mathbf{R}(\theta, \mathbf{n}) = \mathbf{I_3} + \frac{\sin\theta}{\theta}\boldsymbol{\omega}^{\wedge} + \frac{1-\cos\theta}{\theta^2}(\boldsymbol{\omega}^{\wedge})^2 \qquad (3)$$

- $\theta = ||\boldsymbol{\omega}||_2$ – angle, $\mathbf{n} = \frac{\boldsymbol{\omega}}{||\boldsymbol{\omega}||_2}$ – axis, if $|\boldsymbol{\omega}| \neq 0$;
- $\mathbf{R}(0, \mathbf{n}) = \mathbf{I}_{3\times 3} \quad \forall \mathbf{n}$
- $\mathbf{R}(\theta, \mathbf{n}) = \mathbf{R}(-\theta, -\mathbf{n})$
- $\mathbf{R}(\pi + \theta, \mathbf{n}) = \mathbf{R}(\pi - \theta, -\mathbf{n}) \implies$ any rotation can be represented by a unique angle $\theta \in [0, \pi]$ and a unit vector $\mathbf{n}$

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SO(3) log map: Rodrigues (baseline)

$$\begin{aligned}\log : \mathbf{SO}(3) &\mapsto \mathfrak{so}(3)\\ \mathbf{R}_{3\times 3} &\mapsto \boldsymbol{\omega} \in B_\pi(\mathbf{0})\end{aligned}$$

Inverse of the exponential map from Rodrigues' formula (3) [Blanco, 2010]:

$$\theta = \arccos\left(\frac{1}{2}(tr(\mathbf{R}) - 1)\right) \quad (4a)$$

$$\log_{\mathrm{Ro}}(\mathbf{R}) \equiv \boldsymbol{\omega} = \frac{\theta}{2\sin\theta}\left[\mathbf{R} - \mathbf{R}^\top\right]^\vee \quad (4b)$$

- if θ is close to 0 $\implies$ $\frac{\theta}{\sin\theta}$ by Taylor expansion,
- But what if θ is close or equal to π?

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SO(3) topology

Topology of SO(3) rotations group [Hall, 2003]

The Lie group SO(3) is diffeomorphic to the ball in $\mathbb{R}^3$ of radius π with antipodal surface points identified.

$$\mathbf{R}_{3\times 3} \longleftrightarrow \boldsymbol{\omega} \in B_\pi(\mathbf{0}) \tag{5a}$$

$$\mathbf{R}(\pi, \mathbf{n}) = \mathbf{R}(\pi, -\mathbf{n}), \quad \forall \mathbf{n} \tag{5b}$$

SO(3) log map: Rodrigues (baseline)

If $\theta = \pi$ exactly, we can insert it into Rodrigues' formula (3) and get:

$$\frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|} = \mathbf{n} = \left(\epsilon_1 \sqrt{\frac{1}{2}(1 + R_{11})}, \epsilon_2 \sqrt{\frac{1}{2}(1 + R_{22})}, \epsilon_3 \sqrt{\frac{1}{2}(1 + R_{33})} \right)^\top \tag{6}$$

where the individual signs $\epsilon_i = \pm 1$ (if $n_i \neq 0$) are determined up to an overall sign (since $\mathbf{R}(\pi, \mathbf{n}) = \mathbf{R}(\pi, -\mathbf{n})$) via the following relation:

$$\epsilon_i \epsilon_j = \frac{R_{ij}}{\sqrt{(1 + R_{ii})(1 + R_{jj})}}, \quad \text{for } i \neq j, R_{ii} \neq -1, R_{jj} \neq -1$$

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SO(3) log map: S-matrix (around π)

$$\log_S(\mathbf{R}) \equiv \theta \mathbf{n} \tag{7}$$

We can determine the axis of rotation $\mathbf{n}$ for angle $\theta = \arccos\left(\frac{1}{2}(tr(\mathbf{R}) - 1)\right)$ without numerical issues when θ is close to π. Let us define a matrix [Haber, 2011]:

$$S \equiv \mathbf{R} + \mathbf{R}^\top + (1 - tr\mathbf{R})\mathbf{I_3} \tag{8}$$

Then from the Rodrigues' equation (3) in coordinate form:

$$n_i = \epsilon_i \sqrt{\frac{S_{ii}}{3 - tr(\mathbf{R})}}, \tag{9a}$$

$$\epsilon_i = \text{sign}_{\geq 0}([\mathbf{R} - \mathbf{R}^T]_i^\vee), \quad \text{if } \theta \neq \{0, \pi\} \tag{9b}$$

$$\epsilon_j \epsilon_k = \text{sign}_{\geq 0}(S_{jk}), \quad \text{otherwise} \tag{9c}$$

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SO(3) log map: Matrix-to-Quaternion

$$\log_{\mathrm{Q}}(\mathbf{R}) \equiv \boldsymbol{\omega}(q(\mathbf{R})) \tag{10}$$

Mapping from SO(3) matrix to SU(2) unit quaternion is well-defined everywhere due to case differentiation [Shoemake, 1987]:

$$\mathbf{R}_{3\times 3} \mapsto q = (q_s, \mathbf{v})$$

$$\text{if } (\mathrm{tr}(\mathbf{R}) > 0) : t := \sqrt{1 + \mathrm{tr}(\mathbf{R})}, q_s = \frac{t}{2}, \mathbf{v} = \frac{1}{2t}(\mathbf{R} - \mathbf{R}^\top)^\vee \tag{11a}$$

$$\text{else:} \quad t := \sqrt{R_{a,a} - R_{b,b} - R_{c,c} + 1}, \quad q_s = \frac{1}{2t}(R_{c,b} - R_{b,c}), \tag{11b}$$

$$v_a = \frac{t}{2}, \quad v_b = \frac{1}{2t}(R_{b,a} + R_{a,b}), \quad v_c = \frac{1}{2t}(R_{c,a} + R_{a,c}),$$

$$\text{with} \quad a = \arg\max_{i \in \{1,2,3\}} \{R_{i,i}\}, b = (a+1) \bmod 3, c = (a+2) \bmod 3$$

SO(3) log map: Quaternion-to-Axis-angle

$$\log_{\mathrm{Q}}(\mathbf{R}) \equiv \boldsymbol{\omega}(q(\mathbf{R}))$$

$$q = (q_s, \mathbf{v}) = \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\mathbf{n}\right) \mapsto \boldsymbol{\omega} = \theta\mathbf{n}$$

Quaternion to axis-angle (by half-angle formula and non-negativeness of scalar part of the quaternion $q_s \geq 0$):

$$\boldsymbol{\omega} = 4\arctan\left(\frac{|\mathbf{v}|}{\operatorname{sign}_{\geq 0}(q_s)q_s + \sqrt{q_s^2 + |\mathbf{v}|^2}}\right) \cdot \frac{\operatorname{sign}_{\geq 0}(q_s)\mathbf{v}}{|\mathbf{v}|} \quad (12)$$

because $q = -q$ correspond to the same rotation.

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Quaternion to axis-angle (by reciprocal arguments formula):

$$\boldsymbol{\omega} = 2\arctan\left(\frac{|\mathbf{v}|}{q_s}\right)\frac{\mathbf{v}}{|\mathbf{v}|} = \left(\operatorname{sign}_{\geq 0}(q_s)\pi - 2\arctan\left(\frac{q_s}{|\mathbf{v}|}\right)\right)\frac{\mathbf{v}}{|\mathbf{v}|} \tag{13}$$

Extensions of log map out of SO3 manifold

We have defined 3 different implementations of log map:

- Inverse Rodrigues: $\overline{\log}_{\mathrm{Ro}}(\mathbf{R})$ diverges around $\theta = \pi$.
- Defining S-matrix: $\overline{\log}_{\mathrm{S}}(\mathbf{R})$ numerically stable around $\theta = \pi$, but non-differentiable for rotations with zero component of rotation axis $n_i = 0$, diverges around $\theta = 0$.
- Through intermediate quaternion: $\overline{\log}_{\mathrm{Q}}(\mathbf{R})$ numerically stable everywhere, non-differentiable at $\theta = \pi$, but we can choose one-sided derivative.

The line over functions $\overline{f}$ means that we consider these implementations as defined slightly over the SO(3) manifold, such that we can differentiate them in a numerical sense $\dfrac{d\overline{f}(\mathbf{R})}{d\mathbf{R}}$.

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Jacobians of log map extensions

For each extension of the log map we derive analytical Jacobians in a numerical sense (by perturbing components of rotation matrix), using the following assumptions:

- In piece-wise functions we take the one-sided derivative of the chosen function;

e.g. with this assumption we have: $\text{sign}'_{\geq 0}(x) \equiv 0$

$$J_{\overline{\log}_{\text{Ro}}}(\mathbf{R}) = \frac{d\overline{\log}_{\text{Ro}}(\mathbf{R})}{d\mathbf{R}} \in \mathbb{R}^{3\times 9} \tag{14a}$$

$$J_{\overline{\log}_{\text{S}}}(\mathbf{R}) = \frac{d\overline{\log}_{\text{S}}(\mathbf{R})}{d\mathbf{R}} \in \mathbb{R}^{3\times 9} \tag{14b}$$

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Jacobians of log map extensions: observation

Observation 1

Jacobians of extended logarithmic maps are different, even for the rotations with intermediate angles

$$J_{\overline{\log}_{\mathrm{Ro}}}(\mathbf{R}) \neq J_{\overline{\log}_{\mathrm{S}}}(\mathbf{R}) \neq J_{\overline{\log}_{\mathrm{Q}}}(\mathbf{R})$$

Question

Which of the following extensions is correct?

Jacobian of log map on manifold

Definition of Jacobian of function $f : \mathcal{M} \mapsto \mathcal{N}$ acting on manifold [Solà et al., 2020], $\dim(\mathcal{M}) = m, \dim(\mathcal{N}) = n$:

$$\frac{df(\mathcal{X})}{d\mathcal{X}} = \lim_{\mathbf{x} \to 0} \frac{f(\mathcal{X} \boxplus \mathbf{x}) \boxminus f(\mathcal{X})}{\mathbf{x}} \in \mathbb{R}^{n \times m} \tag{15}$$

In this canonical way the Jacobian of log map becomes:

$$\frac{d\log(\mathbf{R})}{d\mathbf{R}(\mathbf{x})} := \lim_{\mathbf{x} \to 0} \frac{\log(\mathbf{R} \boxplus \mathbf{x}) - \log(\mathbf{R})}{\mathbf{x}} = \tag{16a}$$

$$= J_r^{-1}(\boldsymbol{\omega} = \log(\mathbf{R})) = \mathbf{I_3} + \frac{1}{2}\boldsymbol{\omega}^\wedge + \left(\frac{1}{\theta^2} - \frac{1 + \cos\theta}{2\theta \sin\theta}\right)(\boldsymbol{\omega}^\wedge)^2$$

Jacobians of log map on manifold via chain rule

We can represent the canonical Jacobian of log map via chain rule using the Jacobian of extended log map:

$$J_r^{-1}(\mathbf{R}) := \frac{d\log(\mathbf{R})}{d\mathbf{R}(\mathbf{x})} = \tag{17a}$$

$$= \frac{d\overline{\log}(\mathbf{R})}{d\mathbf{R}} \frac{d(R \boxplus \mathbf{x})}{d\mathbf{x}}\bigg|_{\mathbf{x}=\mathbf{0}} =: J_{\overline{\log}}(\mathbf{R}) J_{R\boxplus\cdot}(\mathbf{0})$$

Since we know the Jacobian of the exponential map [Gallego and Yezzi, 2013] and the Jacobian of composition [Blanco, 2010], we can evaluate the Jacobian of box-plus (at any point) $J_{R\boxplus\cdot}(\mathbf{x}) = \frac{d(R \boxplus \mathbf{x})}{d\mathbf{x}}(\mathbf{x})$.

Jacobians of log map on manifold via chain rule: observations

Observation 2

After the matrix multiplication all the Jacobians become equal:

$$\begin{aligned} J_r^{-1}(\mathbf{R}) &= J_{\overline{\log}_{\mathrm{Ro}}}(\mathbf{R}) J_{R\boxplus\cdot}(\mathbf{0}) = \\ &= J_{\overline{\log}_{\mathrm{S}}}(\mathbf{R}) J_{R\boxplus\cdot}(\mathbf{0}) = J_{\overline{\log}_{\mathrm{Q}}}(\mathbf{R}) J_{R\boxplus\cdot}(\mathbf{0}) \end{aligned} \tag{18a}$$

Observation 3

Moreover, for arbitrary point $\mathbf{x} \in B_\pi(\mathbf{0})$:

$$J_{\overline{\log}_{\mathrm{Ro}}}(\mathbf{R}\boxplus\mathbf{x}) J_{R\boxplus\cdot}(\mathbf{x}) = J_{\overline{\log}_{\mathrm{S}}}(\mathbf{R}\boxplus\mathbf{x}) J_{R\boxplus\cdot}(\mathbf{x}) = J_{\overline{\log}_{\mathrm{Q}}}(\mathbf{R}\boxplus\mathbf{x}) J_{R\boxplus\cdot}(\mathbf{x}) \tag{19}$$

Jacobians of log map on manifold via chain rule: failures

The equality of chain rules from observations 2 and 3 hold for all rotations except:

- Jacobian $J_{\overline{\log}_{\mathrm{Ro}}}(\mathbf{R})$ as well as the log map $\overline{\log}_{\mathrm{Ro}}(\mathbf{R})$ of Rodrigues-extension diverges when the rotation angle θ is close to π.
- Jacobian $J_{\overline{\log}_{\mathrm{S}}}(\mathbf{R})$ as well as the log map $\overline{\log}_{\mathrm{S}}(\mathbf{R})$ of S-matrix-extension diverges when the rotation angle θ is close to 0.
- Non-differentiability of $\overline{\log}_{\mathrm{S}}(\mathbf{R})$ S-matrix-extension at the components of rotation axis equal to zero, i.e. $n_i = 0$, nullifies the rows of the Jacobian: $\left(J_{\overline{\log}_{\mathrm{S}}}(\mathbf{R})\right)_{i,:} \equiv 0$.

Theoretical justification

Proposition [Nurlanov, 2021]

Given the functions
$h : X \to Z, g : X \to Y, f : Y \to Z, \quad h(x) = f(g(x)), \quad \forall x \in X,$
g and h are differentiable on X. The intermediate set $Y \subset \mathbb{R}^m$ is such that any open neighbourhood $U(y_0)$ of any point $y_0 \in Y$ does not belong to the set Y.
Let us define an extension of intermediate function f on open superset of Y, such that the extension $\overline{f}_i$ is differentiable at every point of Y. Let us assume that we have 2 differentiable extensions with different Jacobians: $J_{\overline{f}_1}(g(a)) \neq J_{\overline{f}_2}(g(a))$.
Then the result of the chain rule does not depend on the extension as long as the extension is differentiable:

$$J_{\overline{f}_1}(g(a))J_g(a) = J_{\overline{f}_2}(g(a))J_g(a) = J_h(a)$$

Conclusion

- Explored different implementations (extensions) of SO(3) log map: Rodrigues, S-matrix, Quaternion.
  $\Longrightarrow$ Use always log map through intermediate quaternion - avoid degeneracies.
- Derived Jacobian of extended log map for optimization algorithms and proved its correctness.
  $\Longrightarrow$ The use of the chain rule with local parameterization in optimization problems on manifolds in frameworks like Ceres is reasonable as long as the extension of the function beyond the manifold is differentiable.

Contributions to open-source and links

- PR to Sophus library: https://github.com/strasdat/Sophus
- PR to SE(3) tutorial: https://github.com/jlblancoc/tutorial-se3-manifold
- Experiments and report: https://github.com/nurlanov-zh/so3_log_map
- Proof of the proposition: https://math.stackexchange.com/q/4101093/860081

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