A Probabilistic Level Set Formulation for Interactive Organ Segmentation

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ABSTRACT

Level set methods have become increasingly popular as a framework for image segmentation. Yet when used as a generic segmentation tool, they suffer from an important drawback: Current formulations do not allow much user interaction. Upon initialization, boundaries propagate to the final segmentation without the user being able to guide or correct the segmentation. In the present work, we address this limitation by proposing a probabilistic framework for image segmentation which integrates input intensity information and user interaction on equal footings. The resulting algorithm determines the most likely segmentation given the input image and the user input. In order to allow a user interaction in real-time during the segmentation, the algorithm is implemented on a graphics card and in a narrow band formulation.

Keywords: Segmentation, level set, interactive, probabilistic, user interaction, graphics processor unit (GPU), MR, CT, ultrasound, medical imaging

1. INTRODUCTION

The level set method was introduced by Osher and Sethian[1] as a means to implicitly propagate hypersurfaces $C(t)$ in a domain $\Omega \subset \mathbb{R}^n$ by evolving an appropriate embedding function $\phi : \Omega \times [0, T] \to \mathbb{R}$, where:

$$C(t) = \{x \in \Omega \mid \phi(x, t) = 0\}. \tag{1}$$

The ordinary differential equation propagating explicit boundary points is thus replaced by a partial differential equation modeling the evolution of a higher-dimensional embedding function.* The key advantages of this approach are well-known: Firstly, the implicit boundary representation does not depend on a specific parameterization, during the propagation no control point regridding mechanisms need to be introduced. Secondly, evolving the embedding function allows to elegantly model topological changes such as splitting and merging of the embedded boundary. Thirdly, the implicit representation (1) naturally generalizes to hypersurfaces in three or more dimensions. To impose a unique correspondence between a contour and its embedding function one can constrain ϕ to be a signed distance function, i.e. $|\nabla\phi| = 1$ almost everywhere.

The first applications of the level set method to image segmentation were pioneered in the early 90's by Malladi et al.,[3] by Caselles et al.,[4] by Kichenassamy et al.[5] and by Paragios and Deriche.[6] Level set implementations of the Mumford-Shah functional[7] were independently proposed by Chan and Vese[8] and by Tsai et al..[9]

In contrast to other competitive segmentation methods such as graph cut approaches[10, 11] or the Random Walker,[12] the level set method is based on a precise notion of a shape given by the representation in (1). As a consequence, it is amenable to the introduction of statistical shape information which was shown to drastically improve the segmentation of familiar objects in a given image.[13–20] Yet, existing level set methods are inferior to alternative approaches when used as an interactive segmentation tool. Provided an initial boundary, the

*A precursor of the level set method was proposed by Dervieux and Thomasset.[2]

underlying energy minimization propagates these boundaries to the final segmentation. If this segmentation is not the desired one, the user typically cannot modify the outcome. Yet, such a user interaction is highly desirable in many areas of image segmentation. In the field of medical image analysis, for example, the correction of a level set based segmentation by a human expert requires a tedious postprocessing step. In contrast to the methods in,[10, 12] it is not possible for the user to specify areas which are known to be part of the object or of the background.

At the same time, user interaction in the context of image segmentation is typically incorporated on the basis of a simple rule-based reasoning: if a point is marked as object, then it is object, if it is marked as background, then it is background.

In the following, we propose a probabilistic formulation of user interaction in the context of image segmentation. The key idea is to treat the user input as an independent measurement or observation of the true underlying scene. In this sense, the user input is treated on equal footings with the intensity input. The segmentation process subsequently computes the most likely segmentation given the image input and given the user input. In contrast to the rule-based reasoning, this approach allows for inaccuracies in the user input: If a location is marked as 'object' by the user then areas in its vicinity are associated with a certain probability of being part of the object. The respective variational principle thus incorporates the user input as a soft constraint rather than a hard constraint. It is merely another source of information beyond the one given by the input image. In the resulting segmentation process, the evolving boundary is driven by a competition of the image intensity information and the user input.

The formulation allows the use of a narrow band level set scheme. Furthermore, an implementation of our algorithm on a standard graphics card allows for a runtime which is sufficient to provide a smooth interaction of the human expert: An immediate response enables the user to guide the segmentation process interactively.

It should be pointed out that interactive level set methods and fast implementations using Graphics Hardware have been pioneered by Lefohn et al.[21] The notion of interaction in the latter works is, however, restricted to allowing the user to *initialize* the level set evolution. It does not enable the user to further influence the segmentation process or to correct the final segmentation. In fact, the user input does not directly enter the level set evolution, it is merely used to estimate the parameters of a simple intensity model. The key contribution of the present paper is to develop a statistical framework for a user interaction which does not merely provide an initialization for the algorithm, but which actually gives an indication (beyond the intensity information contained in the input image) as to which areas are likely to be part of the object or the background. As a consequence, the evolving boundaries are directly driven by both the intensity information and by the user labeling.

2. USER INTERACTION VIA BAYESIAN INFERENCE

User input as 'measurement'.

Image segmentation can be formulated as a problem of Bayesian inference, see for example.[22, 23] In the context of level set segmentation, for example, the goal of segmentation is to compute the most likely embedding function ϕ given an input image $I : \Omega \to \mathbb{R}$ by maximizing the conditional probability

$$\mathcal{P}(\phi \,|\, I) \propto \mathcal{P}(I \,|\, \phi)\; \mathcal{P}(\phi). \tag{2}$$

The first term in the product allows the introduction of an appropriate image formation model, stating which intensity observations are likely given a particular object-background configuration. The second term allows to impose a prior on the embedding function which stating which segmentations are *a priori* more or less likely. Beyond generic smoothness priors (favoring a short boundary of the segmentation), it has become popular to impose higher-level statistical priors which allow to constrain the shape of the segmentation to a distribution of familiar shapes.

Let us assume we are given an input from a user who marked certain image locations as "object" or as "background". How should such an additional information be incorporated in the above probabilistic formulation? Clearly one cannot treat this additional information as a prior in order to constrain the segmentation process according to the user specifications: This user input is not an *a priori* information because it is clearly dependent on the current segmentation task. In fact, we argue that it should be treated as another observation beyond the input intensity image. This will be detailed in the following for the segmentation of a single object of interest in the image plane Ω. The object of interest may be multiply connected and the segmentation shall be defined in terms of a level set function $\phi : \Omega \to \mathbb{R}$, where image locations x with $\phi(x) \geq 0$ denote parts of the object and locations x with $\phi(x) < 0$ denote the background.

Let us assume, we are given an image $I : \Omega \to \mathbb{R}$ and a user input

$$L : \Omega \to \{-1, 0, 1\}, \tag{3}$$

where the label values reflect the user input, namely:

$$L(x) = \begin{cases} +1, & x \text{ marked as 'object'}, \\ -1, & x \text{ marked as 'background'}, \\ 0, & x \text{ not marked.} \end{cases} \tag{4}$$

The segmentation by an embedding function is computed by maximizing the *a posteriori* probability

$$\mathcal{P}(\phi \,|\, I,\, L) = \frac{\mathcal{P}(I,\, L \,|\, \phi)\, \mathcal{P}(\phi)}{\mathcal{P}(I,\, L)}. \tag{5}$$

with respect to the embedding function ϕ. For the partitioning of the image, merely the sign of the function ϕ is of interest. To impose a unique correspondence between segmentation and embedding function, we additionally contrain ϕ to the space of signed distance functions, i.e. the magnitude of ϕ is given by the distance of the respective point to the boundary. We enforce this constraint with the help of a partial differential equation as detailed in.[24]

In the following, we will expand the Bayesian approach (5) by making various assumptions regarding the relations of intensity information, label information and segmentation. Note that the denominator in (5) does not depend on the segmentation, it can therefore be neglected in the optimization. Firstly, we will assume that the input image and the user-specified labeling are independent, i.e.

$$\mathcal{P}(I,\, L \,|\, \phi) = \mathcal{P}(I \,|\, \phi)\, \mathcal{P}(L \,|\, \phi). \tag{6}$$

This assumption is obviously not accurate since the user input is based on the intensity image. We nevertheless make this assumption in order to derive a computationally simple solution to the segmentation problem. A more accurate modeling of the dependency between intensity image and user labeling may lead to more powerful segmentation tools.

The key challenge is now to model the likelihood $\mathcal{P}(L \,|\, \phi)$ for a user label configuration L given a segmentation ϕ. More precisely, this expression can be written as:

$$\mathcal{P}(L \,|\, \phi) = \prod_{x \in \Omega} \prod_{y \in \Omega} \left[\mathcal{P}(L(x), x \,|\, \phi(y), y)\right]^{dx\, dy}, \tag{7}$$

where we have assumed that the label values at different locations are mutually independent (leading to a product over all x), and that the values of ϕ at different locations are mutually independent (leading to a product over all y). The exponent $dx\, dy$ gives the volume of the bin and guarantees the appropriate continuum limit.

Non-local interaction.

Each individual term $\mathcal{P}(L(x), x \,|\, \phi(y), y)$ models the likelihood of a label value $L(x)$ at location x given the segmentation $\phi(y)$ at location y. We will model this as:

$$\mathcal{P}(L(x), x \,|\, \phi(y), y) \propto \exp\Big(\frac{\nu_2}{2}\, L(x) \text{sign}\big(\phi(y)\big) k_\sigma(x, y)\Big), \tag{8}$$

The reasoning behind this choice for the likelihood is the following: If the level set function at a location y is positive, i.e. $\text{sign}(\phi(y)) = 1$, then a point x in its vicinity is more likely to be marked object ($L(x) = 1$) and less likely to be marked background ($L(x) = -1$), and vice versa for background points. The Gaussian kernel function

$$k_\sigma(x, y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{(x-y)^2}{2\sigma^2}\right)$$

ensures that this coupling decays with the distance between x and y. Our model for the user interaction has two free parameters ν_2 and σ which can be interpreted as follows. The parameter ν_2 provides the overall weight of the user interaction and will determine how strongly the user input will affect the segmentation. The parameter σ defines the spatial range within which a point labeled as object or as background will affect the segmentation. It can therefore be interpreted as a 'brush size'. More sophisticated models for the user interaction are certainly conceivable.

Treatment of Multiple Clicks.

In a practical application, it is very likely that a user will click the same point multiple times. As with most modeling choices in this paper, we will focus on one of the simplest ways to integrate this in the probabilistic formulation. Namely, we assume that all these clicks are independent (as if given by several independent users). This would lead to several label functions $\{L_1, \ldots, L_n\}$. Since these are treated as independent, this amounts to setting a single label function to

$$L = (\#\text{ of positive clicks}) - (\#\text{ of negative clicks}). \tag{9}$$

More sophisticated means of integrating multiple clicks are conceivable. For example, one could drop the independency assumption, stating that if a user clicks a certain point again then this implies he/she is really sure about the selection, such that the effect of a click will be stronger depending on whether it was clicked before. This would lead to a nonlinear (e.g. exponential) dependency of L on the number of clicks.

The intensity model.

We will expand the image term in (6) in a similar fashion:

$$\mathcal{P}(I \mid \phi) = \prod_{x \in \Omega} \left[\mathcal{P}\big(I(x) \mid \phi(x)\big)\right]^{dx}. \tag{10}$$

As above, we assume that intensities at different locations are mutually independent, but in contrast to (7), we additionally assume that the intensity at a location x merely depends on the value of the level set function at that same location. As a specific intensity model, we will assume that object and background intensities are independent samples from two distributions p_1 and p_2:

$$\mathcal{P}\big(I(x) \mid \phi(x)\big) = \begin{cases} p_1(I), & \text{if } \phi(x) \geq 0 \\ p_2(I), & \text{if } \phi(x) < 0 \end{cases} \tag{11}$$

The two distributions can be infered from the respective regions by fitting parametric distributions[25] or more generally by performing a Parzen-Rosenblatt kernel density estimate. We refer to[26] for details.

As a prior on the embedding function ϕ in expression (5), we shall merely impose a constraint on the length of its zero crossing:

$$\mathcal{P}(\phi) = \exp\left(-\nu_1 \int |\nabla H(\phi)|\, dx\right), \tag{12}$$

where H denotes the Heaviside step function. Similar boundary length constraints are common in variational segmentation methods, such as the Snakes[27] or the Mumford-Shah functional.[7] This particular level set formulation is due to Chan and Vese.[8] Given prior knowledge about the shape of objects of interest, more sophisticated statistical shape priors can be imposed (e.g.[13–16, ?–16]).

3. VARIATIONAL FORMULATION

Maximizing the *a posteriori* distribution (5) is equivalent to minimizing its negative logarithm. Up to a constant the latter is given by an energy of the form:

$$E(\phi) = E_{image}(\phi) + \nu_2 \, E_{user}(\phi) + \nu_1 \, E_{shape}(\phi), \tag{13}$$

where the data term is given by the negative logarithm of (10):

$$\begin{aligned} E_{image}(\phi) = & -\int \log p_1\big(I(x)\big) \, H\big(\phi(x)\big) \, dx \\ & -\int \log p_2\big(I(x)\big) \, (1-H(\phi(x))) \, dx. \end{aligned} \tag{14}$$

The energy associated with the user input is given by:

$$\begin{aligned} E_{user}(\phi) &= -\frac{1}{2} \iint L(x) \, \mathrm{sign}\big(\phi(y)\big) \, k_\sigma(x,y) \, dx \, dy \\ &= -\frac{1}{2} \int L_\sigma(y) \, \mathrm{sign}\big(\phi(y)\big) \, dy, \end{aligned} \tag{15}$$

with the Gaussian-smoothed label function

$$L_\sigma(y) = \int L(x) \, k_\sigma(x,y) \, dx.$$

The geometric prior (12) amounts to the length constraint

$$E_{shape}(\phi) = \int |\nabla H(\phi)| \, dx.$$

Using the fact that the sign function can be expressed by the Heaviside function as $\mathrm{sign}(\phi) = 2H(\phi) - 1$, we can combine the three energies to obtain a total energy of the form:

$$\begin{aligned} E(\phi) = & -\int \log p_1(I) H(\phi) dx \\ & -\int \log p_2(I) \big(1-H(\phi)\big) dx \\ & +\int L_\sigma H(\phi) + |\nabla H(\phi)| \, dx, \end{aligned} \tag{16}$$

Minimization of this energy can be done by evolving the following descent equation:

$$\frac{\partial \phi}{\partial t} = \delta(\phi) \Big(\nu_1 \, \mathrm{div} \left(\frac{\nabla \phi}{|\nabla \phi|} \right) + (e_2 - e_1) \, + \, \nu_2 \, L_\sigma \Big), \tag{17}$$

where we introduced the energy densities $e_i = -\log p_i$. The three terms in the above evolution equation can be interpreted as follows: The first term (weighted by ν_1) aims at minimizing the length of the segmenting boundary. The second term drives the boundary to separate the two intensity distributions (infered during the segmentation process by kernel density estimation). And the last term (weighted by ν_2) imposes the user input, driving the segmentation process to favor the segmentation of object and background as indicated by the user.

4. EFFICIENT IMPLEMENTATION VIA NARROW BAND METHODS AND HARDWARE SPEEDUP

We will now discuss the implementation of the above interactive level set segmentation method. The key aspect is to provide for a fast implementation such that the user interaction can be introduced in real-time during the segmentation process, i.e. the segmentation will adapt to the user input without delay. We believe that such direct feedback is important for our method to be pratically useful. The two components allowing for fast implementation will be detailed in the following.

Due to the Delta function multiplying the entire expression (17), we only need to update ϕ values in the vicinity of the zero crossing. This is the central idea of the so-called narrow band methods.[28] In our case, it guarantees that the runtime complexity of our method remains linear in the number of boundary pixels. This advantage of our method will give rise to fast computation times, in particular when segmenting 3D volumes.

Secondly, the key computational challenge is a fairly simple update of the level set values over many iterations. For such large-scale iterative methods, the implementation of respective algorithms on graphics hardware has become increasingly popular, not only because graphics processing units are becoming increasingly powerful (by now allowing floating point computations rather than previous integer compuations), but also because at the current rate their speed tends to double every six month.[29] In order to guarantee maximal speed of our implementation without sacrificing computational accuracy, we reverted to an implementation on a standard graphics card.

In practice, the segmentation of an image or volume of interest is obtained in the following manner:

- The user selects areas of object and of background, using the mouse and some fixed brush size σ.
- The embedding function ϕ implementing the boundary is initialized according to this user input.
- The following three fractional steps are iterated until convergence:
  1. Recompute the distributions p_1 and p_2 by performing a kernel density estimate of the intensity distributions associated with the inside and the outside region.
  2. Evolve the embedding function ϕ according to equation (17), using a fast narrow-band formulation implemented on a graphics card.
  3. Check for new user input from the mouse and update the labeling $L_\sigma(x)$ if necessary.

5. EXPERIMENTAL RESULTS

In Figures 1-3, we present respective steps from the applications of our interactive level set method to the segmentation of MR, CT and ultrasound images in 2 and 3 dimensions. These show that medical structures like the Corpus Calosum and brain turmors (Fig. 1, left side), or various chambers of the heart and the aorta (Fig. 3) can be segmented with a few user clicks. Points which were clicked as 'object' are shown in white, 'background' labels are shown in black. Figure 1, right side, shows the case of a user with large uncertainty (small value of ν_1, large value of σ) - the black labeling therefore affects the segmentation in a rather imprecise (blurred) manner. Figure 2 shows the aorta segmented (with a single click) in a 64x64x64 volume within 8.5 seconds.

6. CONCLUSION

We presented a probabilistic formulation of level set segmentation which allows to integrate a user input on equal footings with the input intensity image. Both are treated as independent observations of an underlying scene. The segmentation process computes the most probable segmentation given the intensity image and given a labeling specifying points which a user marked as 'object' or as 'background'. Our formulation is based on the assumption that user input and intensity information are independent, the effect of the user input is characterized by two parameters which model the spatial uncertainty (corresponding to a "brush size") and the overall reliability of the user labeling. We believe that our approach constitutes a first step toward a more sophisticated probabilistic modeling of user input in the context of image segmentation. A narrow band implementation on graphics hardware provides the necessary speed for a smooth interaction and direct feedback from the segmentation process. As a consequence the user can easily influence the segmentation process during the boundary evolution as well as correct the final segmentation.

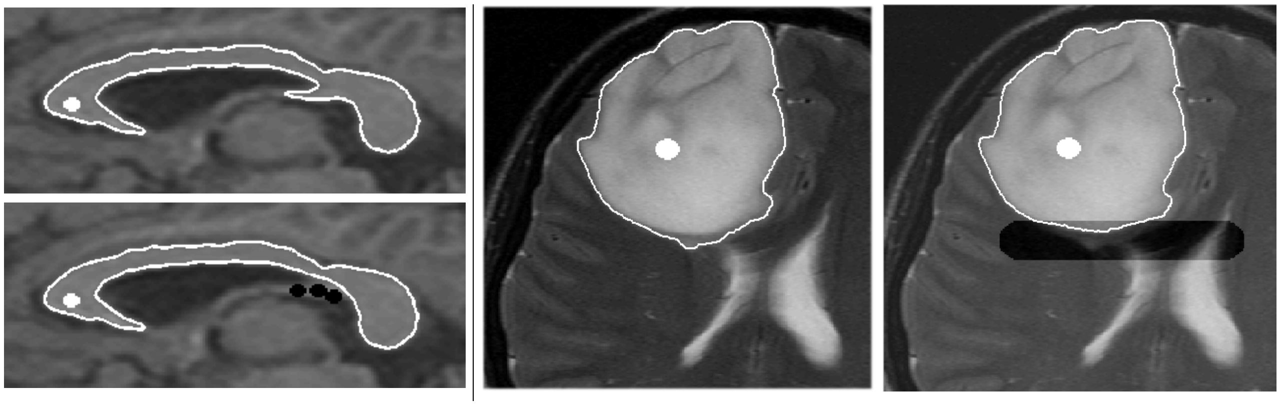

Figure 1. Head MR - Left: A single click gives a segmentation of the Corpus Callosum including a part of the Fornix (small spike on the bottom). The latter is removed by placing three background seeds.
Right: Brain tumor- The use of low weighted seeds allows a rather imprecise user interaction (foreground and background seeds are respectively displayed in white and black).

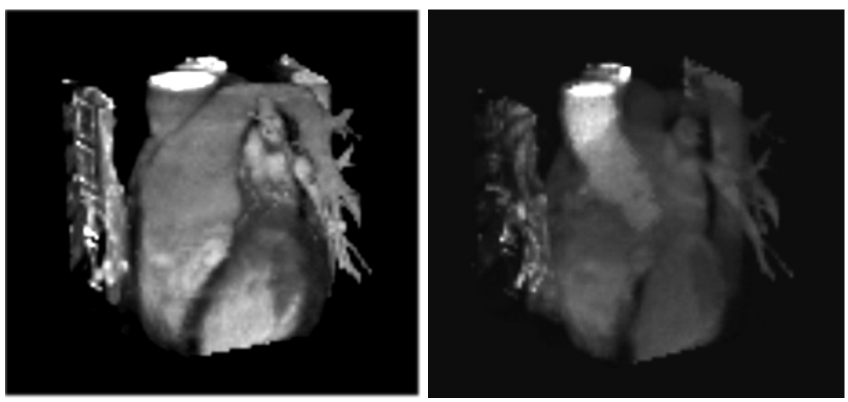

Figure 2. Heart CT - Interactive volume segmentation of the ascending aorta in 8.5 seconds by placing a single seedpoint on a center slice (Image size: 64x64x64).

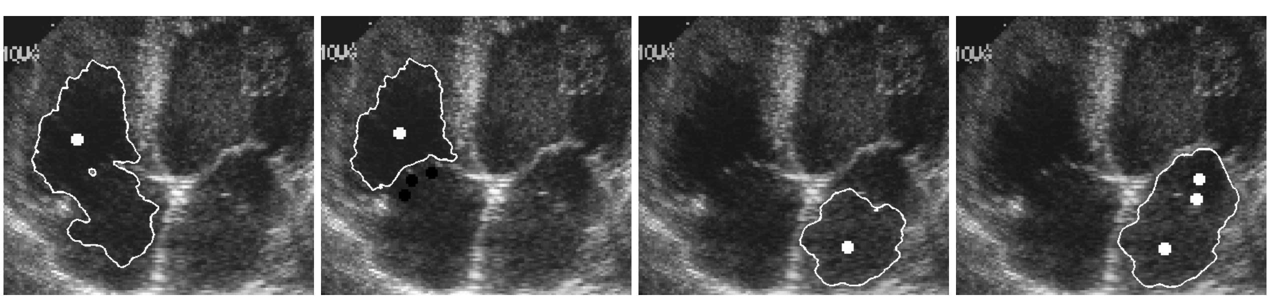

Figure 3. Examples of interactive ultra sound segmentations of two heart chambers with 4 and 3 mouse clicks.

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