Sublabel-Accurate Convex Relaxation of Vectorial Multilabel Energies – Supplementary Material –

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1 Theory

Proof (Proof of Proposition 1). By definition the biconjugate of $\boldsymbol{\rho}$ is given as

$$\begin{aligned} \boldsymbol{\rho}^{**}(\boldsymbol{u}) &= \sup_{\boldsymbol{v}\in\mathbb{R}^{|\mathcal{V}|}} \langle \boldsymbol{u}, \boldsymbol{v}\rangle - \left(\min_{1\le i\le|\mathcal{T}|} \boldsymbol{\rho}_i(\boldsymbol{v})\right)^* \\ &= \sup_{\boldsymbol{v}\in\mathbb{R}^{|\mathcal{V}|}} \langle \boldsymbol{u}, \boldsymbol{v}\rangle - \max_{1\le i\le|\mathcal{T}|} \boldsymbol{\rho}_i^*(\boldsymbol{v}). \end{aligned} \tag{1}$$

We proceed computing the conjugate of $\boldsymbol{\rho}_i$:

$$\begin{aligned} \boldsymbol{\rho}_i^*(\boldsymbol{v}) &= \sup_{\boldsymbol{u}\in\mathbb{R}^{|\mathcal{V}|}} \langle \boldsymbol{u}, \boldsymbol{v}\rangle - \boldsymbol{\rho}_i(\boldsymbol{u}) \\ &= \sup_{\boldsymbol{\alpha}\in\Delta_{n+1}^U} \langle E_i\alpha, \boldsymbol{v}\rangle - \rho\left(T_i\alpha\right), \end{aligned} \tag{2}$$

We introduce the substitution $r := T_i\alpha \in \Delta_i$ and obtain

$$\alpha = K_i^{-1}\begin{pmatrix} r \\ 1 \end{pmatrix}, \quad K_i := \begin{pmatrix} T_i \\ \mathbf{1}^\top \end{pmatrix} \in \mathbb{R}^{n+1\times n+1}, \tag{3}$$

since K_i is invertible for $(\mathcal{V}, \mathcal{T})$ being a non-degenerate triangulation and $\sum_{j=1}^{n+1} \alpha_j = 1$. With this we can further rewrite the conjugate as

$$\begin{aligned} \ldots &= \sup_{\boldsymbol{r}\in\Delta_i} \langle A_i r + b_i, E_i^\top \boldsymbol{v}\rangle - \rho(r) \\ &= \langle E_i b_i, \boldsymbol{v}\rangle + \sup_{r\in\mathbb{R}^n} \langle r, A_i^\top E_i^\top \boldsymbol{v}\rangle - \rho(r) - \delta_{\Delta_i}(r) \\ &= \langle E_i b_i, \boldsymbol{v}\rangle + \rho_i^*(A_i^\top E_i^\top \boldsymbol{v}). \end{aligned} \tag{4}$$

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Proof (Proof of Proposition 2). Define $\boldsymbol{\Psi}_{i,j}$ as

$$\boldsymbol{\Psi}_{i,j}(\boldsymbol{p}) := \begin{cases} \|T_i\alpha - T_j\beta\| \cdot \|\nu\| & \text{if } \boldsymbol{p} = (E_i\alpha - E_j\beta)\nu^\top,\ \alpha,\beta \in \Delta^U_{n+1},\ \nu \in \mathbb{R}^d, \\ \infty & \text{otherwise.} \end{cases} \tag{5}$$

Then, $\boldsymbol{\Psi}$ can be rewritten as a pointwise minimum over the individual $\boldsymbol{\Psi}_{i,j}$

$$\boldsymbol{\Psi}(\boldsymbol{p}) = \min_{1 \le i,j \le |\mathcal{T}|} \boldsymbol{\Psi}_{i,j}(\boldsymbol{p}). \tag{6}$$

We begin computing the conjugate of $\boldsymbol{\Psi}_{i,j}$

$$\begin{aligned} \boldsymbol{\Psi}^*_{i,j}(\boldsymbol{q}) &= \sup_{\boldsymbol{p} \in \mathbb{R}^{d \times |\mathcal{V}|}} \langle \boldsymbol{p}, \boldsymbol{q} \rangle - \boldsymbol{\Psi}_{i,j}(\boldsymbol{p}) \\ &= \sup_{\alpha,\beta \in \Delta^U_{n+1}} \sup_{\nu \in \mathbb{R}^d} \langle Q_i\alpha - Q_j\beta, \nu \rangle - \|T_i\alpha - T_j\beta\| \cdot \|\nu\| \\ &= \sup_{\alpha,\beta \in \Delta^U_{n+1}} \left(\|T_i\alpha - T_j\beta\| \cdot \|\cdot\|\right)^* (Q_i\alpha - Q_j\beta) \\ &= \delta_{\mathcal{K}_{i,j}}(\boldsymbol{q}), \end{aligned} \tag{7}$$

with the set $K_{i,j}$ being defined as

$$\mathcal{K}_{i,j} := \left\{ \boldsymbol{q} \in \mathbb{R}^{d \times |\mathcal{V}|} \;\middle|\; \|Q_i\alpha - Q_j\beta\| \le \|T_i\alpha - T_j\beta\|,\ \alpha,\beta \in \Delta^U_{n+1} \right\}. \tag{8}$$

Since the maximum over indicator functions of sets is equal to the indicator function of the intersection of the sets we obtain for $\boldsymbol{\Psi}^*$

$$\begin{aligned} \boldsymbol{\Psi}^*(\boldsymbol{q}) &= \max_{1 \le i,j \le |\mathcal{T}|} \boldsymbol{\Psi}^*_{i,j}(\boldsymbol{q}) \\ &= \delta_{\mathcal{K}}(\boldsymbol{q}). \end{aligned} \tag{9}$$

Proof (Proof of Proposition 3). Let $\boldsymbol{q} \in \mathbb{R}^{d \times |\mathcal{V}|}$ s.t. $\|Q_i\alpha - Q_j\beta\| \le \|T_i\alpha - T_j\beta\|$ for all $\alpha, \beta \in \Delta^U_{n+1}$ and $1 \le i,j \le |\mathcal{T}|$. For any $1 \le i \le |\mathcal{T}|$ define

$$\begin{aligned} f_i &: \mathbb{R}^n \to \mathbb{R}^n, \\ (\alpha_1, ..., \alpha_n) &\mapsto \sum_{l=1}^n \alpha_l t^{i_l} + (1 - \sum_{l=1}^n \alpha_l) t^{i_{n+1}} = T_i\alpha, \end{aligned} \tag{10}$$

and analogously

$$\begin{aligned} g_i &: \mathbb{R}^n \to \mathbb{R}^{|\mathcal{V}|} \\ (\alpha_1, ..., \alpha_n) &\mapsto \sum_{l=1}^n \alpha_l \boldsymbol{q}^{i_l} + (1 - \sum_{l=1}^n \alpha_l) \boldsymbol{q}^{i_{n+1}} = Q_i\alpha. \end{aligned} \tag{11}$$

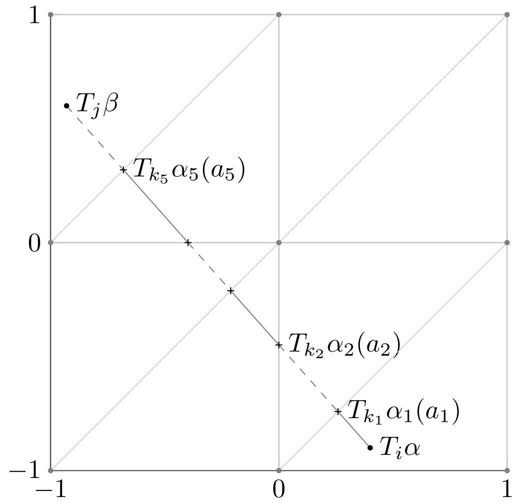

Fig. 1: Figure illustrating the second direction of the proof of Proposition 4. The gray dots and lines visualize the triangulation $(\mathcal{V}, \mathcal{T})$. The line segment between $T_i\alpha$ and $T_j\beta$ is composed of shorter line segments which are fully contained in one of the triangles. On each of the triangles the inequality (15) holds, which allows to conclude that it holds for the whole line segment.

Let us choose an $\alpha \in \mathbb{R}^n$ such that $\alpha_i > 0$, $\sum_l \alpha_l < 1$. Then $\|Q_i\alpha - Q_j\beta\| \leq \|T_i\alpha - T_j\beta\|$ for all $\alpha, \beta \in \Delta_{n+1}^U$ and $1 \leq i, j \leq |\mathcal{T}|$ implies that

$$\|g_i(\alpha) - g_i(\alpha - h)\| \leq \|f_i(\alpha) - f_i(\alpha - h)\|, \tag{12}$$

holds for all vectors h with sufficiently small entries. Inserting the definitions of g_i and f_i we find that

$$\|Q_i Dh\| \leq \|T_i Dh\| \tag{13}$$

holds for all h with sufficiently small entries. For a non-degenerate triangle, $T_i D$ is invertible and a simple substitution yields that

$$\|Q_i D(T_i D)^{-1}\tilde{h}\|_2 \leq \|\tilde{h}\|, \tag{14}$$

holds for all $\tilde{h}$ with sufficiently small entries. This means that the operator norm of $D_{\boldsymbol{q}}^i$ induced by the ℓ^2 norm, i.e. the S^∞ norm, is bounded by one.

Let us now show the other direction. For $\boldsymbol{q} \in \mathbb{R}^{d \times |\mathcal{V}|}$ s.t. $\left\|D_{\boldsymbol{q}}^i\right\|_{S^\infty} \leq 1,\ 1 \leq i \leq |\mathcal{T}|$, note that inverting the above computation immediately yields that

$$\|Q_k\alpha - Q_k\beta\| \leq \|T_k\alpha - T_k\beta\| \tag{15}$$

holds for all $1 \leq k \leq |\mathcal{T}|$, $\alpha, \beta \in \Delta_{n+1}^U$. Our goal is to show that having this inequality on each simplex is sufficient to extend it to arbitrary pairs of simplices. The overall idea of this part of the proof is illustrated in Fig. 1.

Let $1 \leq i, j \leq |\mathcal{T}|$ and $\alpha, \beta \in \mathbb{R}^n$ with $\alpha_l, \beta_l \geq 0$, $\sum_l \alpha_l \leq \sum_l \beta_l \leq 1$ be given. Consider the line segment

$$\begin{aligned} c(\gamma) : [0,1] &\to \mathbb{R}^d \\ \gamma &\mapsto \gamma \, T_j \beta + (1-\gamma) \, T_i \alpha. \end{aligned} \tag{16}$$

Since the triangulated domain is convex, there exist $0 = a_0 < a_1 < \ldots < a_r = 1$ and functions $\alpha_l(\gamma)$ such that for $\gamma \in [a_l, a_{l+1}]$, $0 \leq l \leq r-1$ one can write $c(\gamma) = \gamma \, T_j \beta + (1-\gamma) \, T_i \alpha = T_{k_l} \alpha_l(\gamma)$ for some $1 \leq k_l \leq T$. The continuity of $c(\gamma)$ implies that $T_{k_l} \alpha_l(a_{l+1}) = T_{k_{l+1}} \alpha_{l+1}(a_{l+1})$, i.e. these points correspond to both simplices, k_l and k_{l+1}. Note that this also means that $Q_{k_l} \alpha_l(a_{l+1}) = Q_{k_{l+1}} \alpha_{l+1}(a_{l+1})$. The intuition of this construction is that the $c(a_{l+1})$ are located on the boundaries of adjacent simplices on the line segment. We find

$$\begin{aligned} \|T_i \alpha - T_j \beta\| &= \sum_{l=0}^{r-1} (a_{l+1} - a_l) \|T_i \alpha - T_j \beta\| \\ &= \sum_{l=0}^{r-1} \|(a_{l+1} - a_l)(T_i \alpha - T_j \beta)\| \\ &= \sum_{l=0}^{r-1} \|a_{l+1} T_i \alpha - a_l T_i \alpha - a_{l+1} T_j \beta + a_l T_j \beta\| \\ &= \sum_{l=0}^{r-1} \|a_l T_j \beta + (1 - a_l) T_i \alpha - (a_{l+1} T_j \beta + (1 - a_{l+1}) T_i \alpha)\| \\ &= \sum_{l=0}^{r-1} \|T_{k_l} \alpha_l(a_l) - T_{k_l} \alpha_l(a_{l+1})\| \\ &\overset{(15)}{\geq} \sum_{l=0}^{r-1} \|Q_{k_l} \alpha_l(a_l) - Q_{k_l} \alpha_l(a_{l+1})\| \\ &\geq \left\| \sum_{l=0}^{r-1} (Q_{k_l} \alpha_l(a_l) - Q_{k_l} \alpha_l(a_{l+1})) \right\| \\ &= \left\| \sum_{l=0}^{r-1} (Q_{k_l} \alpha_l(a_l) - Q_{k_{l+1}} \alpha_{l+1}(a_{l+1})) \right\| \\ &= \|Q_{k_0} \alpha_0(a_0) - Q_{k_r} \alpha_r(a_r)\| \\ &= \|Q_i \alpha - Q_j \beta\|, \end{aligned} \tag{17}$$

which yields the assertion.

Proof (Proof of Proposition 4). Let $\Delta = \mathsf{conv}\{t^1, \ldots, t^{n+1}\}$ be given by affinely independent vertices $t^i \in \mathbb{R}^n$. We show that our lifting approach applied to the label space Δ solves the convexified unlifted problem, where the dataterm was replaced by its convex hull on Δ. Let the matrices $T \in \mathbb{R}^{n \times (n+1)}$ and $D \in \mathbb{R}^{(n+1) \times n}$ be defined through

$$T = \left(t^1, \ldots, t^{n+1}\right), \; D = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ -1 & \ldots & -1 \end{pmatrix}, \; TD = \left(t^1 - t^{n+1}, \ldots, t^n - t^{n+1}\right), \tag{18}$$

The transformation $x \mapsto t^{n+1} + TDx$ maps $\Delta_e = \mathsf{conv}\{0, e^1, \ldots, e^n\} \subset \mathbb{R}^n$ to Δ. Now consider the following lifted function $\boldsymbol{u} : \Omega \to \mathbb{R}^{n+1}$ parametrized through $\tilde{u} : \Omega \to \Delta_e$:

$$\boldsymbol{u}(x) = \left(\tilde{u}_1(x), \ldots, \tilde{u}_n(x), 1 - \textstyle\sum_{j=1}^n \tilde{u}_j(x)\right). \tag{19}$$

Consider a fixed $x \in \Omega$. Plugging this lifted representation into the biconjugate of the lifted dataterm $\boldsymbol{\rho}$ yields:

$$\begin{aligned}
\boldsymbol{\rho}^{**}(\boldsymbol{u}) &= \sup_{v \in \mathbb{R}^{n+1}} \langle \boldsymbol{u}, \boldsymbol{v} \rangle - \sup_{\alpha \in \Delta_{n+1}^U} \langle \alpha, \boldsymbol{v} \rangle - \rho(T\alpha) \\
&= \sup_{v \in \mathbb{R}^{n+1}} \left\langle \left(\tilde{u}_1(x), \; \ldots, \; \tilde{u}_n(x), \; 1 - \sum_{j=1}^n \tilde{u}_j(x) \right), \boldsymbol{v} \right\rangle - \\
&\qquad \sup_{\alpha \in \Delta_{n+1}^U} \langle \alpha, \boldsymbol{v} \rangle - \rho(T\alpha) \\
&= \sup_{v \in \mathbb{R}^{n+1}} \langle \tilde{u}, D^\top \boldsymbol{v} \rangle + \boldsymbol{v}_{n+1} - \\
&\qquad \sup_{\alpha \in \Delta_{n+1}^U} \left\langle \left(\alpha_1, \; \ldots, \; \alpha_n, \; 1 - \sum_{j=1}^n \alpha_j \right), \boldsymbol{v} \right\rangle - \\
&\qquad\qquad \rho\left(\sum_{j=1}^n \alpha_j t^j + \left(1 - \sum_{j=1}^n \alpha_j\right) t^{n+1} \right) \\
&= \sup_{v \in \mathbb{R}^{n+1}} \langle \tilde{u}, D^\top \boldsymbol{v} \rangle + \boldsymbol{v}_{n+1} - \sup_{\alpha \in \Delta_{n+1}^U} \boldsymbol{v}_{n+1} + \langle \alpha, D^\top \boldsymbol{v} \rangle - \rho(t^{n+1} + TD\alpha)
\end{aligned} \tag{20}$$

Since $D^\top$ is surjective, we can apply the substitution $\tilde{v} = D^\top \boldsymbol{v}$:

$$\begin{aligned}
\ldots &= \sup_{\tilde{v} \in \mathbb{R}^n} \langle \tilde{u}, \tilde{v} \rangle - \sup_{\alpha \in \Delta_{n+1}^U} \langle \alpha, \tilde{v} \rangle - \rho(t^{n+1} + TD\alpha) \\
&= \sup_{\tilde{v} \in \mathbb{R}^n} \langle \tilde{u}, \tilde{v} \rangle - \sup_{w \in \Delta} \langle (TD)^{-1}(w - t^{n+1}), \tilde{v} \rangle - \rho(w).
\end{aligned} \tag{21}$$

In the last step the substitution $w = t^{n+1} + TD\alpha \Leftrightarrow \alpha = (TD)^{-1}(w - t^{n+1})$ was performed. This can be further simplified to

$$\begin{aligned} \ldots &= \sup_{\tilde{v} \in \mathbb{R}^n} \langle \tilde{u}, \tilde{v} \rangle + \langle (TD)^{-1} t^{n+1}, \tilde{v} \rangle - (\rho + \delta_\Delta)^*((TD)^{-T} \tilde{v}) \\ &= \sup_{\tilde{v} \in \mathbb{R}^n} \langle \tilde{u} + (TD)^{-1} t^{n+1}, \tilde{v} \rangle - (\rho + \delta_\Delta)^*((TD)^{-T} \tilde{v}) \\ &= \sup_{\tilde{v} \in \mathbb{R}^n} \langle TD\tilde{u} + t^{n+1}, (TD)^{-T} \tilde{v} \rangle - (\rho + \delta_\Delta)^*((TD)^{-T} \tilde{v}). \end{aligned} \tag{22}$$

Since TD is invertible we can perform another substitution $v' = (TD)^{-T}\tilde{v}$.

$$\begin{aligned} \ldots &= \sup_{v' \in \mathbb{R}^n} \langle TD\tilde{u} + t^{n+1}, v' \rangle - (\rho + \delta_\Delta)^*(v') \\ &= (\rho + \delta_\Delta)^{**}(t^{n+1} + TD\tilde{u}). \end{aligned} \tag{23}$$

The lifted regularizer is given as:

$$\boldsymbol{R}(\boldsymbol{u}) = \sup_{\boldsymbol{q}: \Omega \to \mathbb{R}^{d \times n+1}} \int_\Omega \langle \boldsymbol{u}, \operatorname{Div} \boldsymbol{q} \rangle - \Psi^*(\boldsymbol{q}) \, \mathrm{d}x \tag{24}$$

Using the parametrization by $\tilde{u}$, this can be equivalently written as

$$\sup_{\boldsymbol{q}(x) \in \mathcal{K}} \int_\Omega \sum_{j=1}^n \tilde{u}_j \operatorname{Div}(\boldsymbol{q}_j - \boldsymbol{q}_{n+1}) + \operatorname{Div} \boldsymbol{q}_{n+1} \, \mathrm{d}x, \tag{25}$$

where the set $\mathcal{K} \subset \mathbb{R}^{d \times n+1}$ can be written as

$$\mathcal{K} = \{ \boldsymbol{q} \in \mathbb{R}^{d \times n+1} \mid \| D^\top \boldsymbol{q}^\top (TD)^{-1} \|_{S^\infty} \leq 1 \}. \tag{26}$$

Note that since $\boldsymbol{q}_{n+1} \in C_c^\infty(\Omega, \mathbb{R}^d)$, the last term $\operatorname{Div} \boldsymbol{q}_{n+1}$ in (25) vanishes by partial integration. With the substituion $\tilde{q}(x) = D^\top \boldsymbol{q}(x)^\top$ we have

$$\sup_{\tilde{q} \in \tilde{\mathcal{K}}} \int_\Omega \langle \tilde{u}, \operatorname{Div} \tilde{q} \rangle \, \mathrm{d}x, \tag{27}$$

with set $\tilde{\mathcal{K}} \subset \mathbb{R}^{d \times n}$:

$$\tilde{\mathcal{K}} = \{ q \in \mathbb{R}^{d \times n} \mid \| q (TD)^{-1} \|_{S^\infty} \leq 1 \}. \tag{28}$$

Note that since $\boldsymbol{q}_i \in C_c^\infty(\Omega, \mathbb{R}^d)$, the same holds for the linearly transformed $\tilde{q}$. With another substituion $q'(x) = \tilde{q}(x)(TD)^{-1}$ we have

$$\begin{aligned} \cdots &= \sup_{q' \in \mathcal{K}'} \int_\Omega \langle \tilde{u}, \operatorname{Div} q' TD \rangle \, \mathrm{d}x \\ &= \sup_{q' \in \mathcal{K}'} \int_\Omega \langle TD\tilde{u}, \operatorname{Div} q' \rangle \, \mathrm{d}x \end{aligned} \tag{29}$$

Fig. 2: Joint estimation of mean and variance. Our formulation can optimize difficult nonconvex joint optimization problems with continuous label spaces.

where the set $\mathcal{K}' \subset \mathbb{R}^{d \times n+1}$ is given as

$$\mathcal{K}' = \{q \in \mathbb{R}^{d \times n} \mid \|q\|_{S^\infty} \leq 1\}, \tag{30}$$

which is the usual unlifted definition of the total variation $TV(t^{n+1} + TD\tilde{u})$.

This shows that the lifting method solves

$$\min_{\tilde{u}:\Omega \to \Delta_e} \int_\Omega (\rho(x, \cdot) + \delta_\Delta)^{**}(t^{n+1} + TD\tilde{u}(x))\mathrm{d}x + \lambda TV(t^{n+1} + TD\tilde{u}), \tag{31}$$

which is equivalent to the original problem but with a convexified data term.

2 Additional Experiment: Adaptive Denoising

In this experiment we jointly estimate the mean μ and variance σ of an image $I : \Omega \to \mathbb{R}$ according to a Gaussian model. The label space is chosen as $\Gamma = [0, 255] \times [1, 10]$ and the dataterm as proposed in [1]:

$$\rho(x, \mu(x), \sigma(x)) = \frac{(\mu(x) - I(x))^2}{2\sigma(x)^2} + \frac{1}{2}\log(2\pi\sigma(x)^2). \tag{32}$$

As the projection onto the epigraph of $(\rho + \delta_\Delta)^*$ seems difficult to compute, we approximate ρ by a piecewise linear function using 29×29 sublabels and convexify it using the quickhull algorithm [2]. In Fig. 2 we show the result of minimizing (32) with total variation regularization.

References

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