Partial Functional Correspondence

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Joint work with


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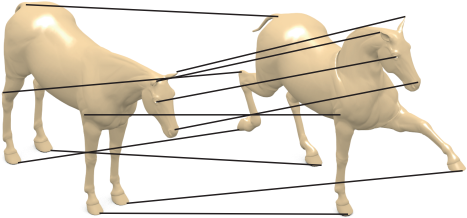
SGP, Berlin, 21 June 2016

Shape correspondence problem

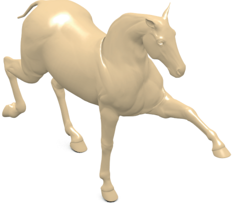


Isometric

Shape correspondence problem

Isometric Partial

Shape correspondence problem



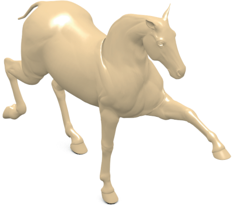
Isometric

Partial

Different representation

Shape correspondence problem

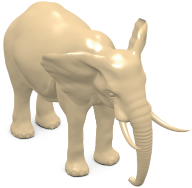
Isometric



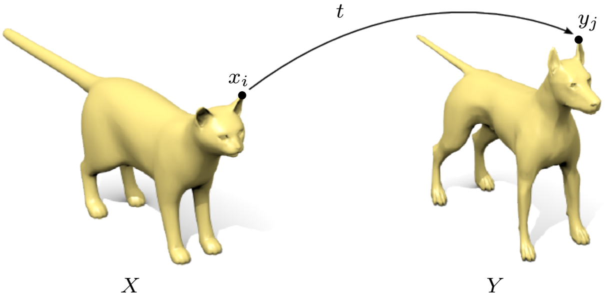
Partial

Different representation

Non-isometric

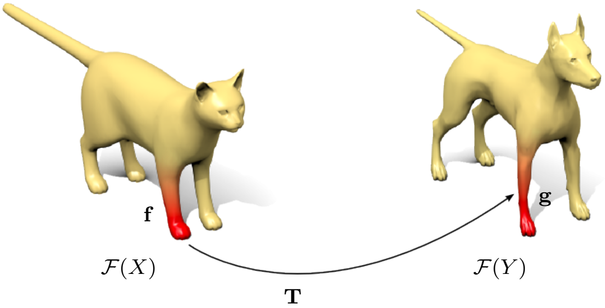


Point-wise maps



Point-wise maps $t\colon X \to Y$

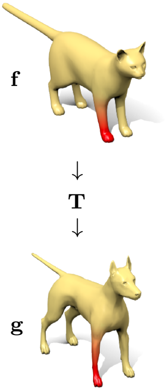
Functional maps



Functional maps $\mathbf{T}\colon \mathcal{F}(X) \to \mathcal{F}(Y)$

Ovsjanikov et al., 2012

Functional correspondence



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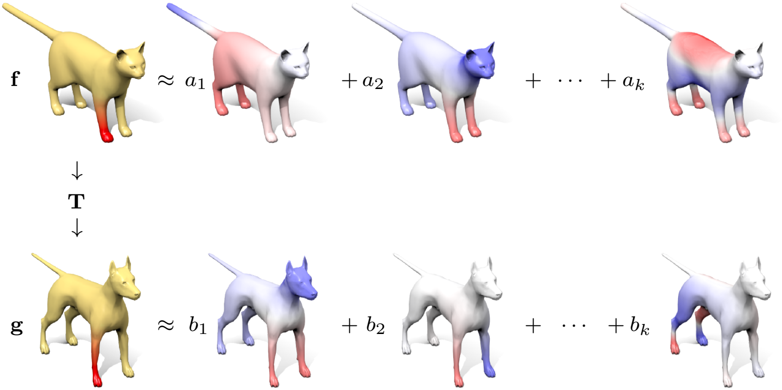

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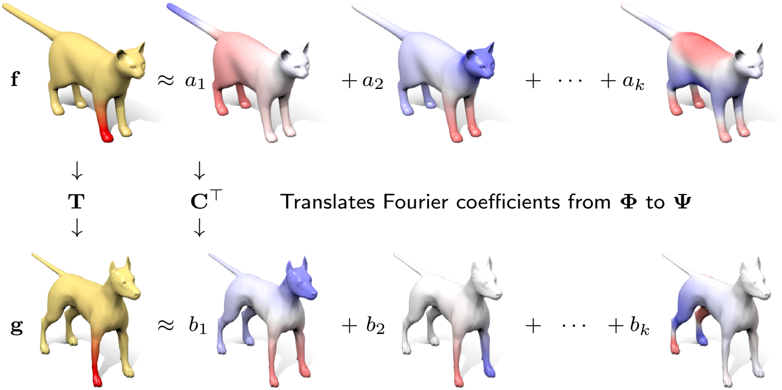

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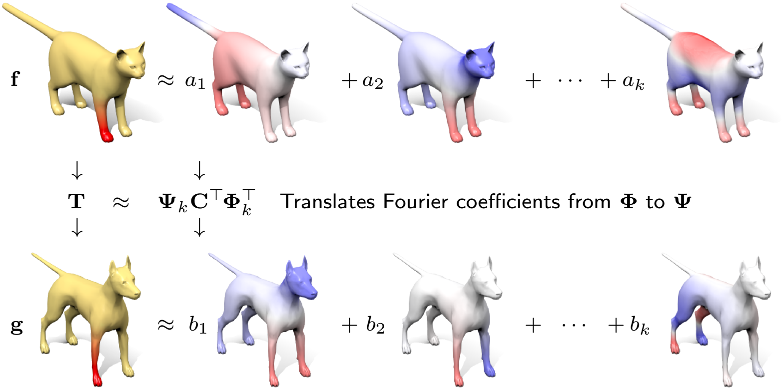
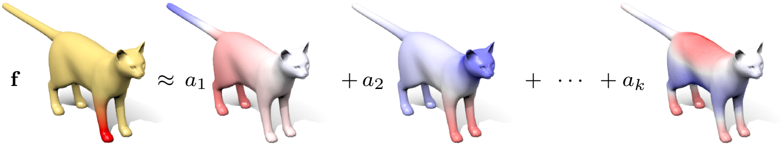


$$\mathbf{g}^\top \mathbf{\Psi}_k = \mathbf{f}^\top \mathbf{\Phi}_k \mathbf{C}$$

where $\mathbf{\Phi}_k = (\phi_1, \ldots, \phi_k)$, $\mathbf{\Psi}_k = (\psi_1, \ldots, \psi_k)$ are Laplace-Beltrami eigenbases

Ovsjanikov et al., 2012

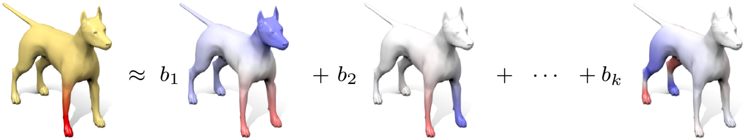
Functional correspondence



$$\downarrow \qquad \downarrow$$

$\mathbf{T} \approx \mathbf{\Psi}_k \mathbf{C}^\top \mathbf{\Phi}_k^\top$ Translates Fourier coefficients from $\mathbf{\Phi}$ to $\mathbf{\Psi}$

$$\downarrow \qquad \downarrow$$



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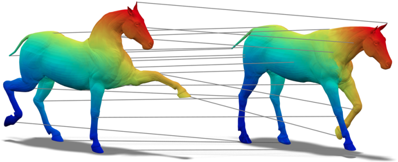
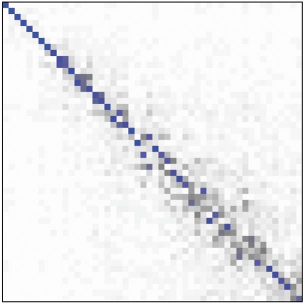
Ovsjanikov et al., 2012

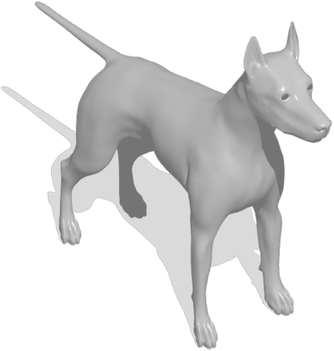
Functional correspondence in Laplacian eigenbases





For isometric simple spectrum shapes $\mathbf{C}$ is diagonal since $\boldsymbol{\psi}_i = \pm \mathbf{T} \boldsymbol{\phi}_i$

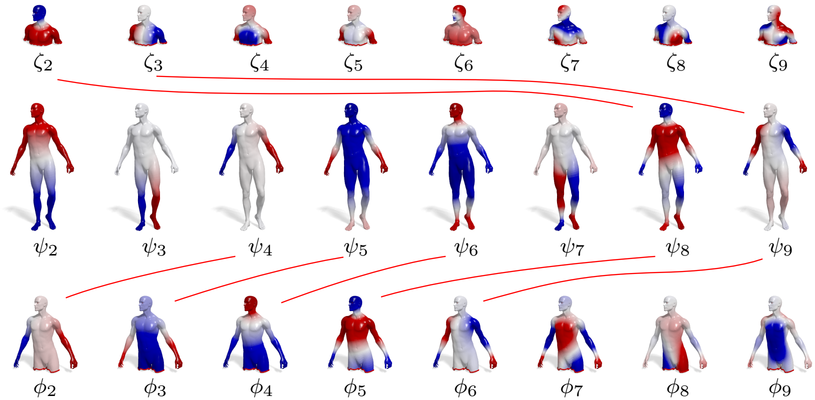
Our setting



Full model

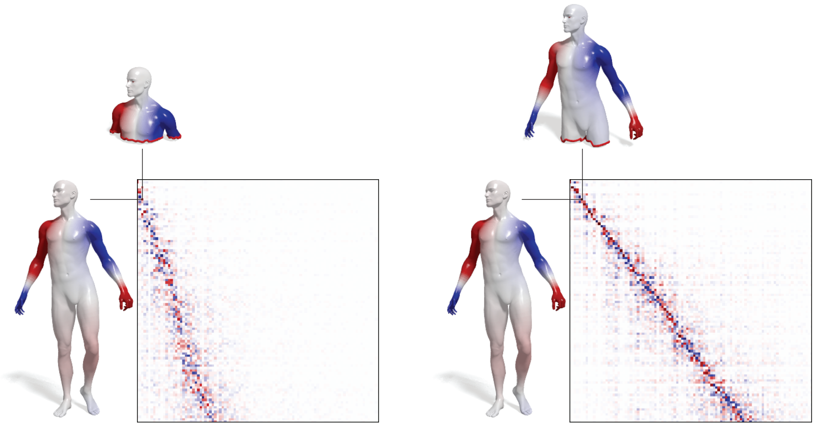
Partial query

Partial Laplacian eigenvectors



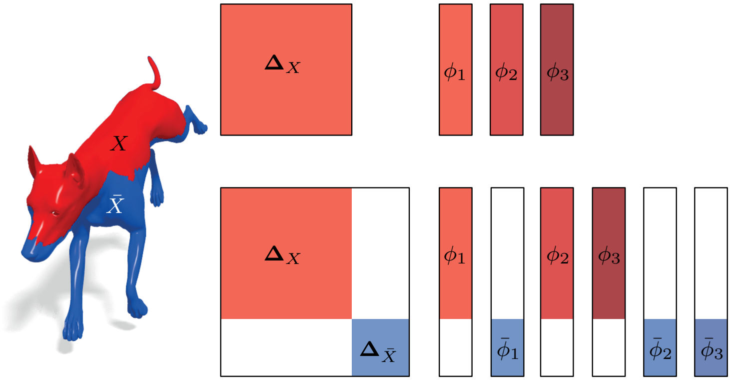
Laplacian eigenvectors of a shape with missing parts
(Neumann boundary conditions)

Partial Laplacian eigenvectors



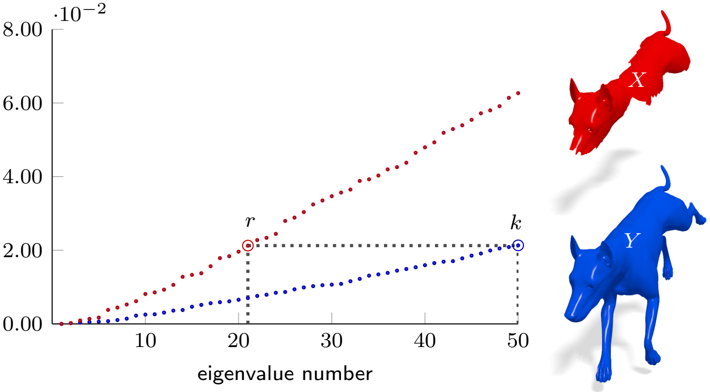
Functional correspondence matrix $\mathbf{C}$

Perturbation analysis: intuition



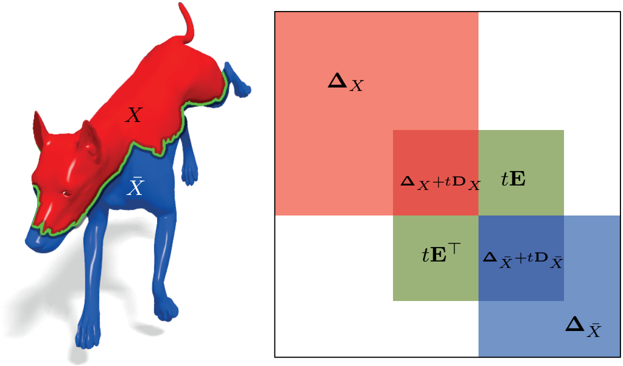
- Ignoring boundary interaction: disjoint parts (block-diagonal matrix)
- Eigenvectors = Mixture of eigenvectors of the parts

Perturbation analysis: eigenvalues



- Slope $\frac{r}{k} \approx \frac{\text{area}(X)}{\text{area}(Y)}$ (depends on the area of the cut)
- Consistent with Weyl's law

Perturbation analysis: details



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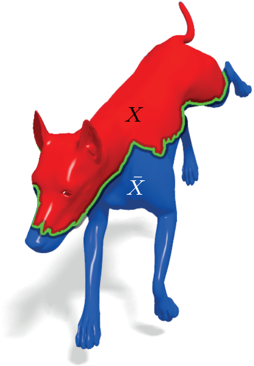
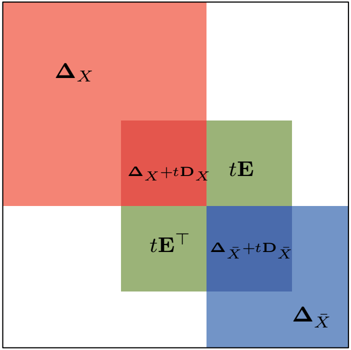
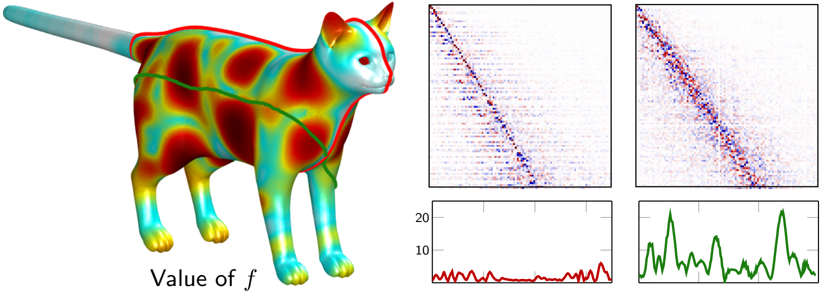




"How would the Laplacian eigenvalues and eigenvectors of the red part change if we attached a blue part to it?"

Perturbation analysis: boundary interaction strength



- Eigenvector perturbation depends on length and position of the boundary
- Perturbation strength $\leq c \int_{\partial X} f(x)dx$, where

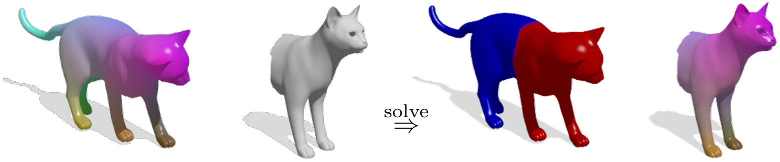
$$f(x) = \sum_{\substack{i,j=1 \\ j \neq i}}^{n} \left(\frac{\phi_i(x)\phi_j(x)}{\lambda_i - \lambda_j} \right)^2$$

Partial functional maps

Given full model shape Y and query shape X corresponding to an unknown approximately isometric part $Y' \subset Y$, the partial functional map $\mathbf{T} : \mathcal{F}(X) \to \mathcal{F}(Y)$ is given by

$$\mathbf{T}\mathbf{f} = \mathrm{diag}(v)\mathbf{g}$$

where $v \in \mathcal{F}(Y)$ is an indicator function of the part



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Optimization problem w.r.t. correspondence and part

$$\min_{\mathbf{C}, v} \|\mathbf{F}^\top \mathbf{\Phi} \mathbf{C} - \mathbf{G}^\top \mathrm{diag}(\eta(v)) \mathbf{\Psi}\|_{2,1} + \rho_{\mathrm{corr}}(\mathbf{C}) + \rho_{\mathrm{part}}(v)$$

where $\eta(t) = \frac{1}{2}(\tanh(2t - 1) + 1)$ saturates the part membership function

Partial functional maps

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- **Part regularization**

$$\rho_{\mathrm{part}}(v) = \mu_1 \left(\mathrm{area}(X) - \int_Y \eta(v) dx \right)^2 + \mu_2 \int_Y \|\nabla_Y \eta(v)\| dx$$

Bronstein and Bronstein 2008

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$$\rho_{\mathrm{corr}}(\mathbf{C}) = \mu_3 \|\mathbf{C} \circ \mathbf{W}\|_{\mathrm{F}}^2 + \mu_4 \sum_{i \neq j} (\mathbf{C}^\top \mathbf{C})_{ij}^2 + \mu_5 \sum_i ((\mathbf{C}^\top \mathbf{C})_{ii} - d_i)^2$$

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Partial functional maps

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Bronstein and Bronstein 2008

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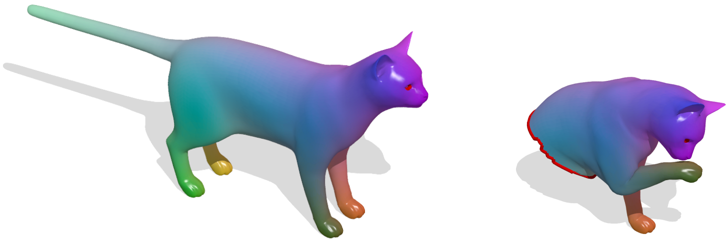
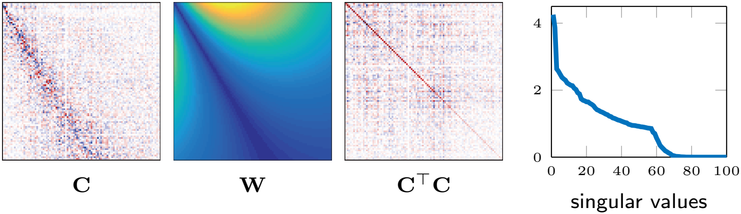
- $\mathbf{F}, \mathbf{G}$ = dense SHOT descriptor in all our experiments

Bronstein and Bronstein 2008; Tombari et al. 2010

Structure of partial functional correspondence





Alternating minimization

- $\mathbf{C}$-**step:** fix v^*, solve for correspondence $\mathbf{C}$

$$\min_{\mathbf{C}} \|\mathbf{F}^\top \mathbf{\Phi} \mathbf{C} - \mathbf{G}^\top \mathrm{diag}(\eta(v^*)) \mathbf{\Psi}\|_{2,1} + \rho_{\mathrm{corr}}(\mathbf{C})$$

- v-**step:** fix $\mathbf{C}^*$, solve for part v

$$\min_{v} \|\mathbf{F}^\top \mathbf{\Phi} \mathbf{C}^* - \mathbf{G}^\top \mathrm{diag}(\eta(v)) \mathbf{\Psi}\|_{2,1} + \rho_{\mathrm{part}}(v)$$

Alternating minimization

- **C-step:** fix v^*, solve for correspondence $\mathbf{C}$

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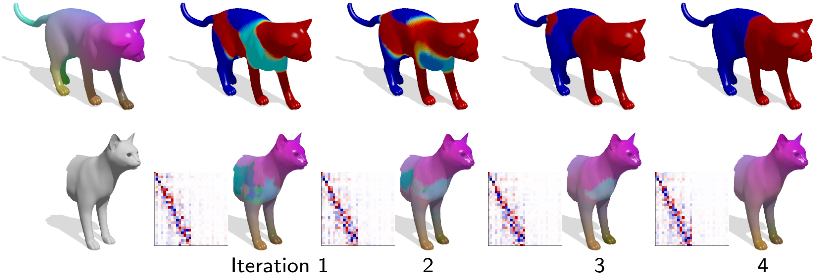
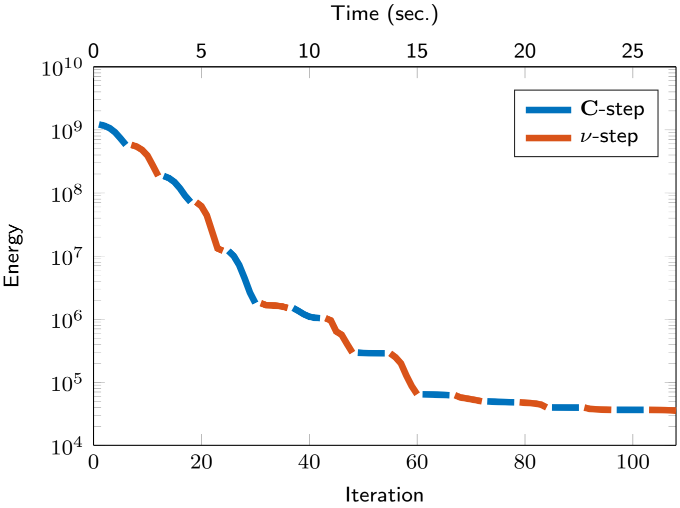


Example of convergence



Examples of partial functional maps

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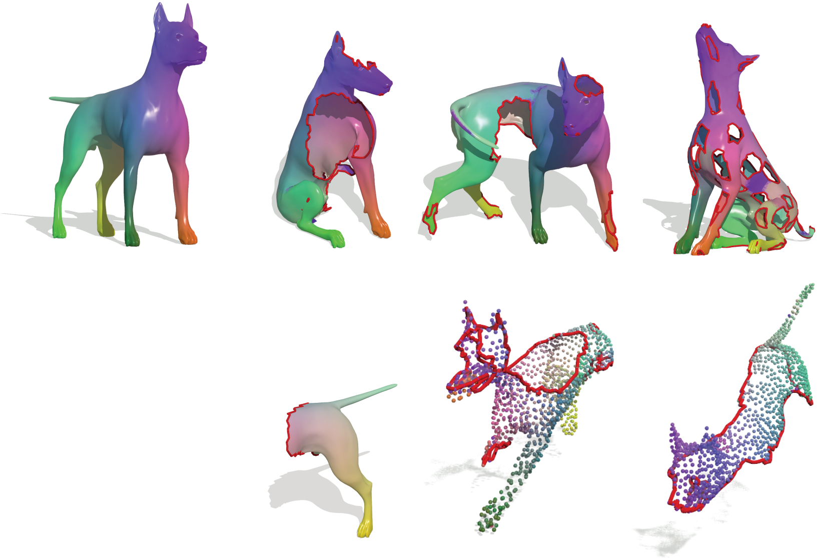

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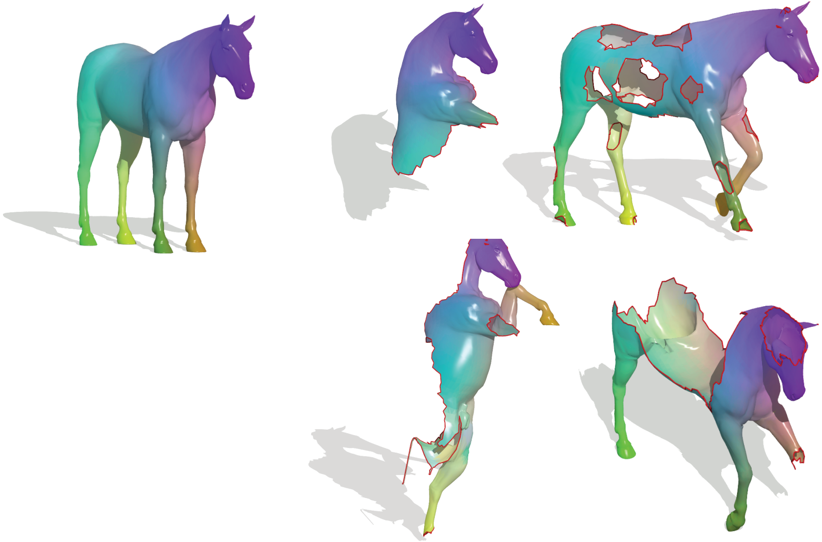
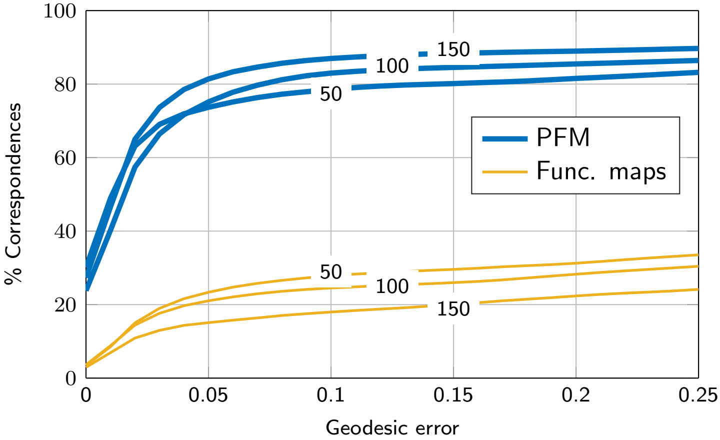
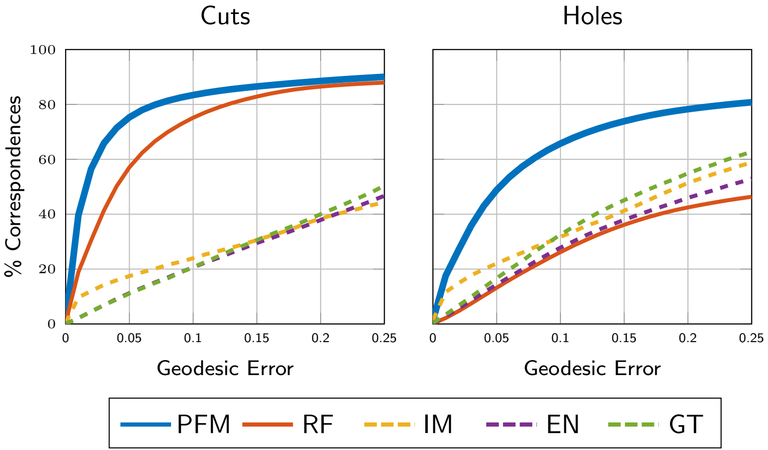


Partial functional maps vs Functional maps



Correspondence performance for different rank values k

Partial correspondence performance (SHREC'16)



SHREC'16 Partial Matching benchmark Rodolà et al. 2016; Methods: Rodolà, Cosmo, Bronstein, Torsello, Cremers 2016 (**PFM**); Sahillioğlu and Yemez 2012 (IM); Rodolà et al. 2012 (GT); Rodolà et al. 2013 (EN); Rodolà et al. 2014 (RF)

Conclusions

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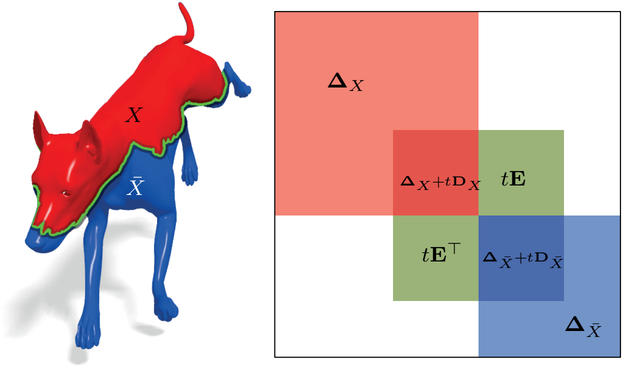
Thank you!

Perturbation analysis: details



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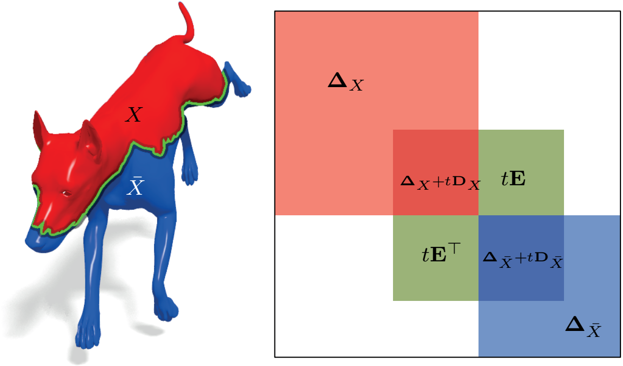

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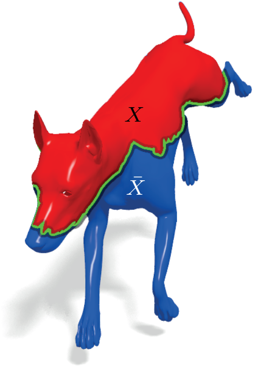

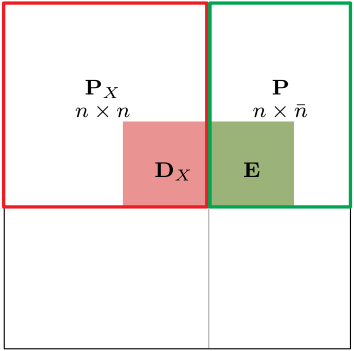

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Perturbation analysis: details

Denote $\mathbf{\Delta}_X + t\mathbf{P}_X = \mathbf{\Phi}(t)\mathbf{\Lambda}(t)\mathbf{\Phi}(t)^\top$, $\mathbf{\Delta}_{\bar{X}} = \bar{\mathbf{\Phi}}\bar{\mathbf{\Lambda}}\bar{\mathbf{\Phi}}^\top$, $\mathbf{\Phi} = \mathbf{\Phi}(0)$, and $\mathbf{\Lambda} = \mathbf{\Lambda}(0)$.

Theorem 1 (eigenvalues) The derivative of the non-trivial eigenvalues is given by

$$\frac{d}{dt}\lambda_i = \boldsymbol{\phi}_i^\top \mathbf{P}_X \boldsymbol{\phi}_i \qquad \mathbf{P}_X = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_X \end{pmatrix}$$

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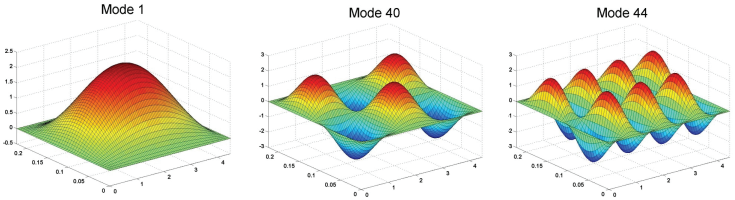
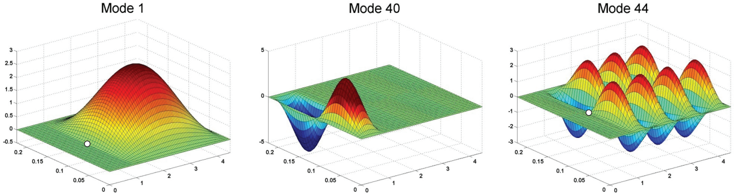
Theorem 2 (eigenvectors) Assuming $\lambda_i \neq \lambda_j$ for $i \neq j$ and $\lambda_i \neq \bar{\lambda}_j$ for all i, j, the derivative of the non-trivial eigenvectors is given by

$$\frac{d}{dt}\boldsymbol{\phi}_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\boldsymbol{\phi}_i^\top \mathbf{P}_X \boldsymbol{\phi}_j}{\lambda_i - \lambda_j}\boldsymbol{\phi}_j + \sum_{j=1}^{\bar{n}} \frac{\boldsymbol{\phi}_i^\top \mathbf{P}\, \bar{\boldsymbol{\phi}}_j}{\lambda_i - \bar{\lambda}_j}\bar{\boldsymbol{\phi}}_j \qquad \mathbf{P} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{E} & \mathbf{0} \end{pmatrix}$$

(bi-)Laplacian perturbation: typical picture



Plate



Punctured plate

Figure: Filoche, Mayboroda 2009