Generalized Roof Duality for Multi-Label Optimization: Optimal Lower Bounds and Persistency

Thomas Windheuser[1,2] Hiroshi Ishikawa[2] Daniel Cremers[1]

[1]Technische Universität München

[2]Waseda University

will be presented at the ECCV 2012 in Florence

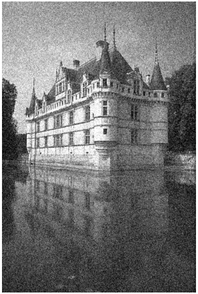
Markov Random Fields in Computer Vision

Image Segmentation

Boykov and Jolly [ICCV 2001]

Markov Random Fields in Computer Vision

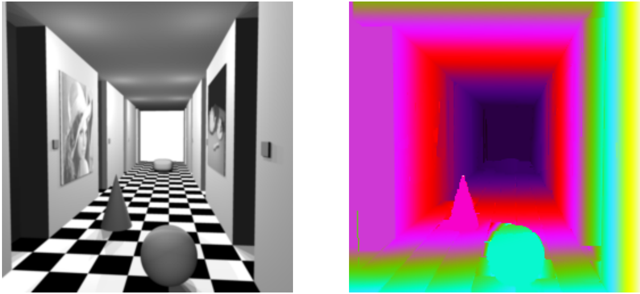
Image Denoising



Roth and Black [CVPR 2005]

Markov Random Fields in Computer Vision

Stereo Reconstruction



Woodford et al. [CVPR, 2008]

General Form

$$\min_{\mathbf{x}\in\mathcal{L}^n} f(\mathbf{x}) = \min_{\mathbf{x}\in\mathcal{L}^n} \sum_{c\in\mathcal{C}} \theta_c(\mathbf{x}_c)$$

- $\mathbf{x} \in \mathcal{L}^n$ set of n variables (vertices, pixels)
- $\mathcal{L}$ finite set of labels
- $\mathcal{C} \subset 2^n$ set of cliques
- $\theta_c : \mathcal{L}^{|c|} \to \mathbb{R}$ potential function
- $\mathrm{order}(f) = \max_{c\in\mathcal{C}} |c| - 1$

General Form

$$\min_{\mathbf{x}\in\mathcal{L}^n} f(\mathbf{x}) = \min_{\mathbf{x}\in\mathcal{L}^n} \sum_{c\in\mathcal{C}} \theta_c(\mathbf{x}_c)$$

Generally NP-Hard!

Related Work - Optimization Strategies

$$\min_{\mathbf{x}\in\mathcal{L}^n} f(\mathbf{x}) = \min_{\mathbf{x}\in\mathcal{L}^n} \sum_{c\in\mathcal{C}} \theta_c(\mathbf{x}_c)$$

- Graph Cut: first-order, binary, submodular
  Kolmogorov and Zabih [PAMI 2004]
- Graph Cut: first-order, multi-label, submodular
  Ishikawa [PAMI 2003] Schlesinger and Flach [TR 2006]
- Alpha Expansion: multi-label, local solution
  Boykov et al. [PAMI 2001]
- QPBO: first-order, binary, partial labeling
  Hammer et al. [Math. Programming 1984]
- Higher-Order Clique Reduction: higher-order, binary
  Ishikawa [CVPR 2009] Fix et al. [ICCV 2011]

Related Work
On Partial Optimality in Multi-label MRFs

On Partial Optimality in Multi-label MRFs
Kohli, Shekhovtsov, Rother, Kolmogorov, Torr [ICML 2008]

- multi-label, first-order, non-submodular functions
- partial labeling

Related Work - Generalized Roof Duality

Generalized Roof Duality

Kahl and Standmark [ICCV 2011, Discrete Appl. Math. 2012]

- binary, higher-order, non-submodular functions
- partial labeling
- optimal submodular relaxation

In This Presentation

Generalized Roof Duality for Multi-Label Optimization
Windheuser, Ishikawa, Cremers [ECCV 2012]

- multi-label, higher-order, non-submodular functions
- partial labeling
- optimal submodular relaxation

Submodular Relaxation

$$\min_{\mathbf{x}\in\mathcal{L}^n} f(\mathbf{x}) \qquad \longrightarrow \qquad \min_{(\mathbf{x},\mathbf{y})\in\mathcal{L}^{2n}} g(\mathbf{x},\mathbf{y})$$

***g* satisfies:**

(A) $\forall \mathbf{x} \in \mathcal{L}^n : f(\mathbf{x}) = g(\mathbf{x}, \overline{\mathbf{x}})$,

(B) g is submodular and

(C) $\forall (\mathbf{x},\mathbf{y}) \in \mathcal{L}^{2n} : g(\mathbf{x},\mathbf{y}) = g(\overline{\mathbf{y}}, \overline{\mathbf{x}})$,

where $\overline{x} = |\mathcal{L}| - 1 - x$.

(A) $\text{image}(f) \subset \text{image}(g)$
Minimizer of g is a lower bound of the minimizer of f.

(B) Global minimum of g can be computed efficiently.

(C) Symmetry condition required for the *Persistency Theorem*.

Optimal Lower Bound

Find the submodular relaxation with the optimal lower bound.

$$
\begin{aligned}
\max_{g} \min_{(\mathbf{x},\mathbf{y}) \in \mathcal{L}^{2n}} \; & g(\mathbf{x}, \mathbf{y}) \\
\text{s. t.} \; & \text{(A) } \forall \mathbf{x} \in \mathcal{L}^{n} : f(\mathbf{x}) = g(\mathbf{x}, \overline{\mathbf{x}}), \\
& \text{(B) } g \text{ is submodular and} \\
& \text{(C) } \forall (\mathbf{x}, \mathbf{y}) \in \mathcal{L}^{2n} : g(\mathbf{x}, \mathbf{y}) = g(\overline{\mathbf{y}}, \overline{\mathbf{x}}).
\end{aligned}
$$

Label Ranges

Definition:

$$\mathcal{S}^n := \{(\mathbf{x}, \mathbf{y}) \in \mathcal{L}^{2n} \mid \forall i \in \{1, \dots, n\} : x_i + y_i < \ell\}.$$

Lemma: For any submodular symmetric function $g : \mathcal{L}^{2n} \to \mathbb{R}$ the following statement is true:

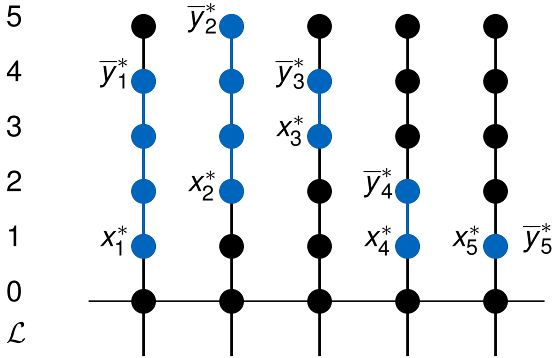
$$\forall \mathbf{x}, \mathbf{y} \in \mathcal{L}^n : g(\mathbf{x}, \mathbf{y}) \geq g(\min(\mathbf{x}, \bar{\mathbf{y}}), \min(\bar{\mathbf{x}}, \mathbf{y})).$$

Since $(\min(\mathbf{x}, \bar{\mathbf{y}}), \min(\bar{\mathbf{x}}, \mathbf{y})) \in \mathcal{S}^n$, there always exists a point in $\mathcal{S}^n$ that minimizes g.

Label Ranges

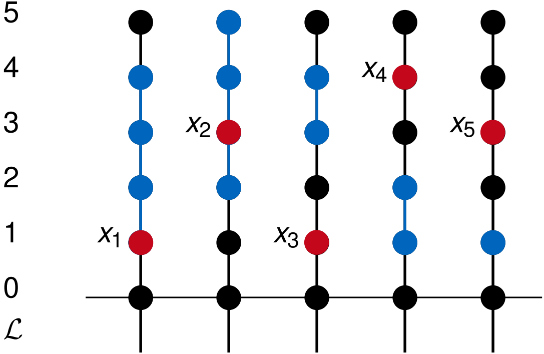
For every g there exists a minimizer $(\mathbf{x}^*, \mathbf{y}^*) \in \arg\min g(\mathbf{x}, \mathbf{y})$ such that

$$\forall i : 0 \leq x_i^* \leq \overline{y}_i^* < |\mathcal{L}|.$$



Overwrite Operator - Projection onto Label Ranges

The *overwrite operator* $\mathbf{x} \leftarrow (\mathbf{x}^*, \mathbf{y}^*)$ projects any point $\mathbf{x} \in \mathcal{L}^n$ onto the ranges defined by $(\mathbf{x}^*, \mathbf{y}^*) \in \mathcal{L}^{2n}$



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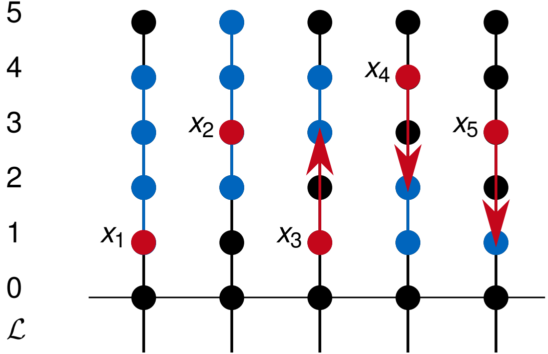

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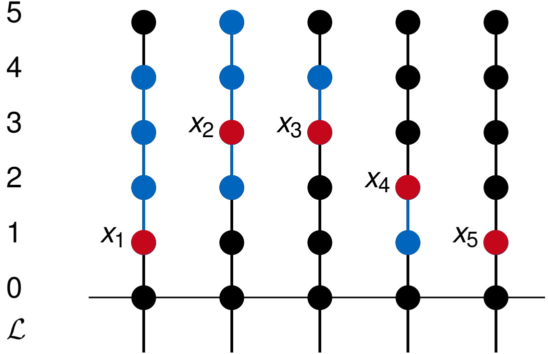


Main Result

Persistency Theorem:

Let g be a function satisfying (A)-(C) and $(\mathbf{x}^*, \mathbf{y}^*) \in \mathcal{L}^{2n}$ be a minimizer of g, then

$$\forall \mathbf{x} \in \mathcal{L}^n : f(\mathbf{x} \leftarrow (\mathbf{x}^*, \mathbf{y}^*)) \leq f(\mathbf{x})$$

In particular, if $\mathbf{x} \in \arg\min(f)$, then also $\mathbf{x} \leftarrow (\mathbf{x}^*, \mathbf{y}^*) \in \arg\min(f)$.

Main Result

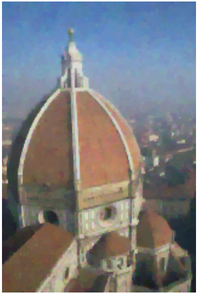
Theorem for First-Order MRF Functions:

For any first-order MRF function there is a closed-form solution to the problem of finding the optimal submodular relaxation.

Denoising Example

original

noisy

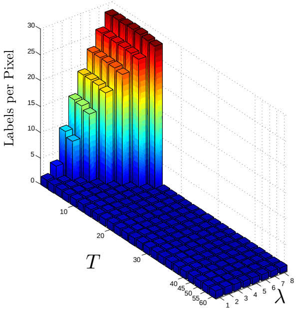


lower bounds

upper bounds

$$\mathbf{x}^* = \arg\min_{\mathbf{x}} \sum_{p\in\mathcal{P}} (\mathcal{I}(p) - \mathbf{x}(p))^2 + \lambda \sum_{pq\in\mathcal{N}} \min(|\mathbf{x}(p) - \mathbf{x}(q)|, T)$$

Denoising Example



original

noisy $\sigma(0.2)$

lower bounds

upper bounds

$T = 8, \lambda = 6$

lower bounds

upper bounds

$T = 10, \lambda = 6$

lower bounds

upper bounds

$T = 12, \lambda = 6$

Conclusion

Submodular Relaxations for Non-Submodular Multi-Label Functions

Proof of the *Persistency Theorem*

- any number of linearly ordered labels
- non-submodular
- potential functions of arbitrary order

First-Order Non-Submodular Functions

- optimal lower bound
- closed-form solution
- partial labeling / ranges of labels