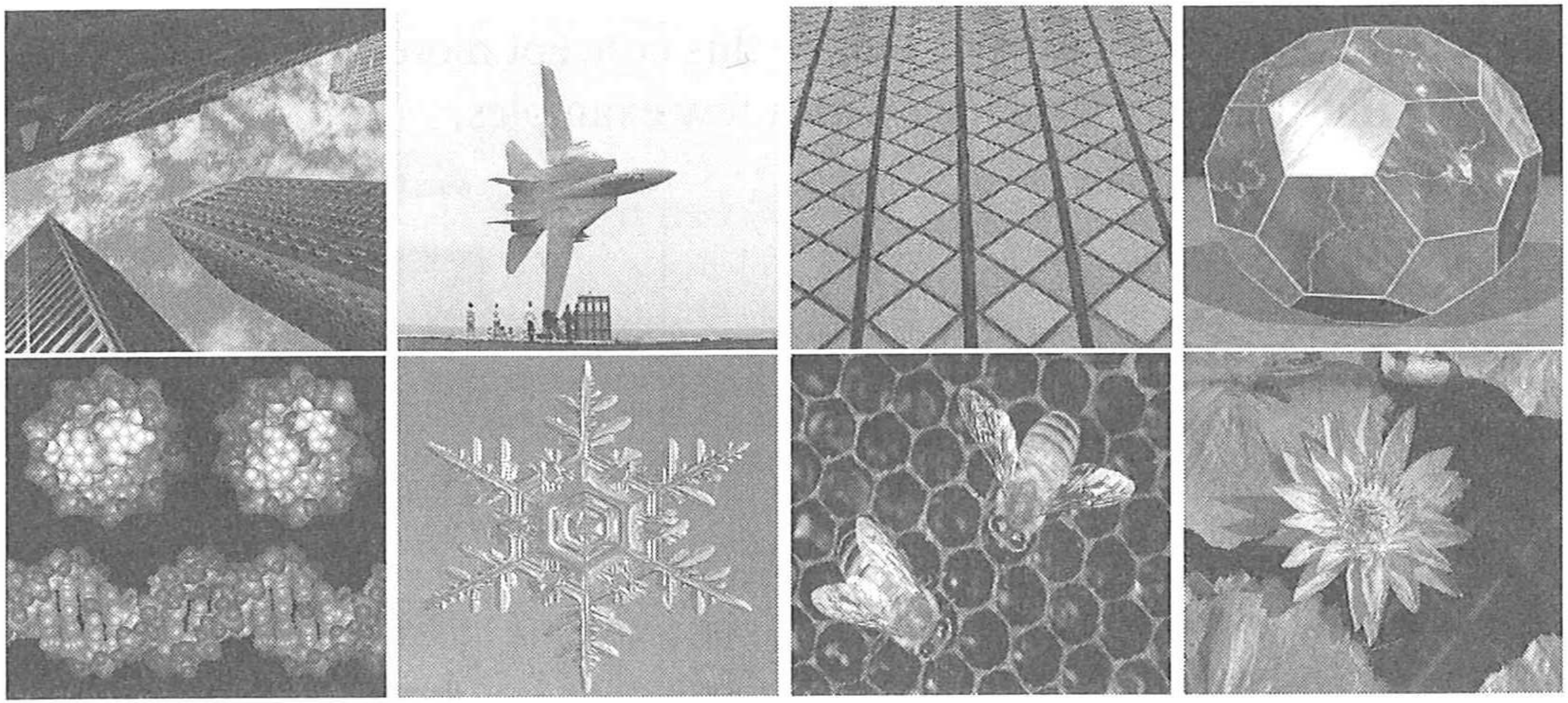
3D Reconstruction From Symmetry

Symmetry in nature and man-made world

Numerous objects in our environment exhibit a certain degree of symmetry. Examples are house facades with repetitive window structure, the tiles of a floor or the honey comb of bees. There exist several vantage points from which these structures appear identical.



Symmetric structures in man-made environments and in nature.
(Source: Vladimir Bulatov)

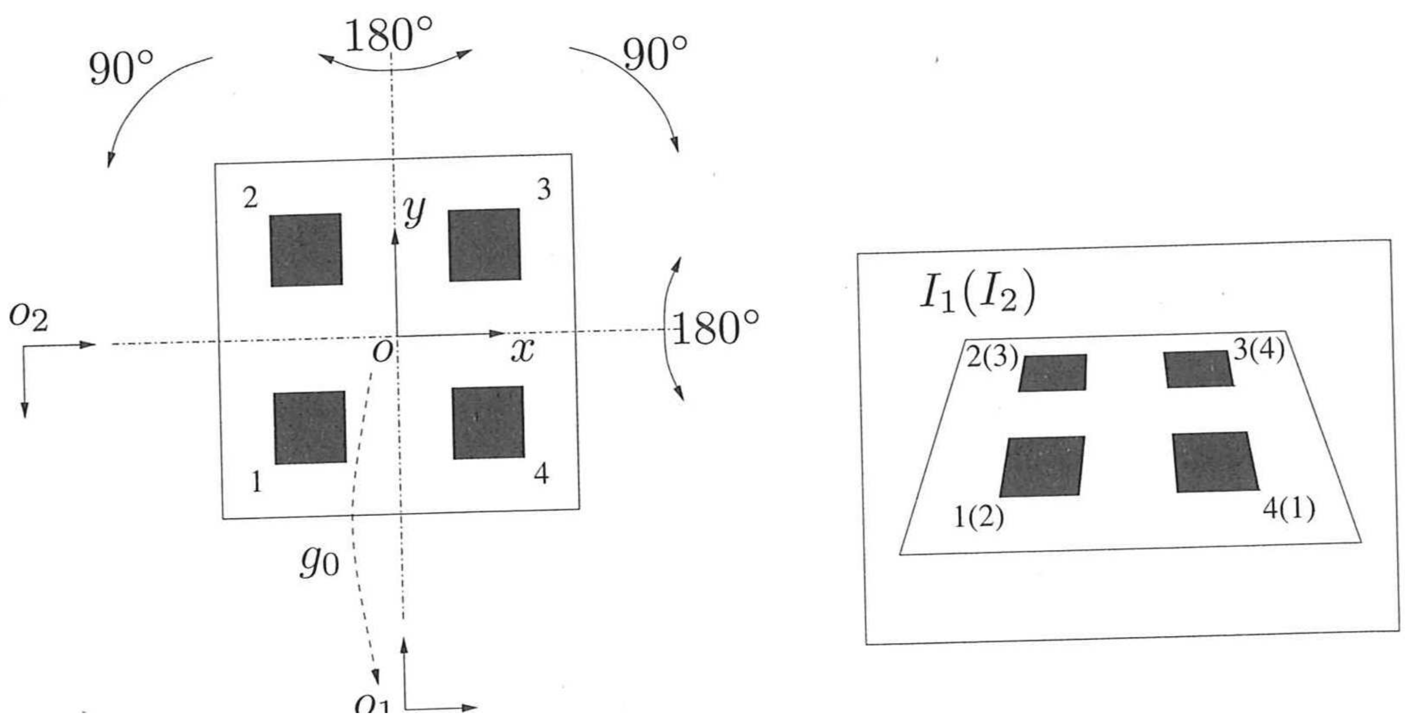
Symmetry and perceptual illusions

distorted room illusion, Adelbert Ames, Jr. (1880-1955).

Human observers heavily rely on symmetry assumptions 3D modeling from a single view. If these assumptions are violated our 3D interpretation may be erroneous.

The checkerboard example

Consider a checkerboard (left) observed from two vantage points, o_1 and o_2. From both points, the observed image would look like the one on the right.



Thus upon rotation around the center point by 90° the observation is unchanged. In addition, reflection along the x- and y-axis leave the observed image unaffected.

Different types of symmetry

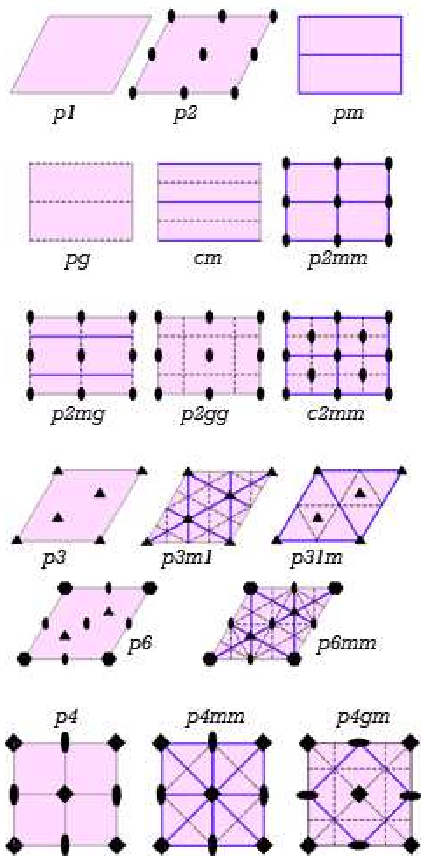
One can distinguish three types of symmetry transformations that may give rise to equivalent views:

- Rotational symmetry: rotations around a specific axis.
- Reflective symmetry: reflection along an axis.
- Translational symmetry: Specific translations.

Fedorov 1885, Weyl 1952: Rotational, reflective and translational symmetries are the only isometric symmetries in Euclidean space, all others are only a combination of these three.

17 ways of tiling a plane

The study of symmetry goes back to the beginnings of scientific studies. The Egyptians already knew all 17 ways of tiling the plane.



17 ways of tiling a plane by translating and rotating a geometric pattern. (Source: MathWorld)

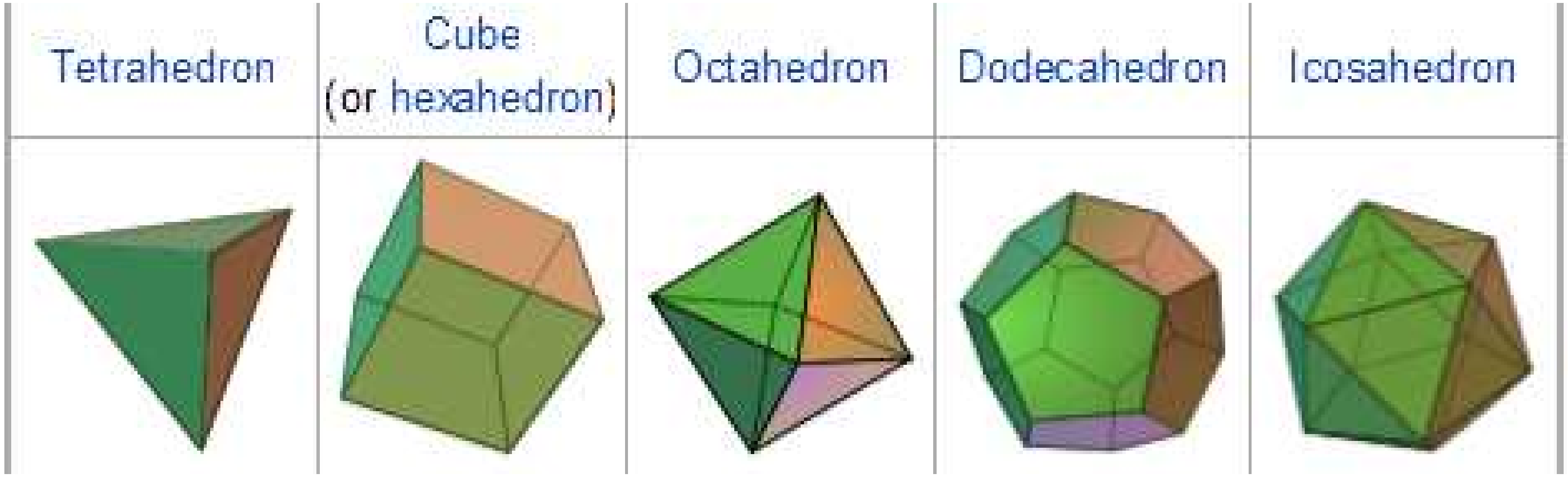
Example: The Alhambra in Spain

Alhambra in Granada, Spain, 14th century.

(Source: Wikipedia)

The platonic solids

Among objects that exhibit a 3D rotational symmetry are the so-called Platonic solids that were already known to Pythagoras.



Five platonic solids, i.e. convex polyhedra which are regular.

(Source: Wikipedia)

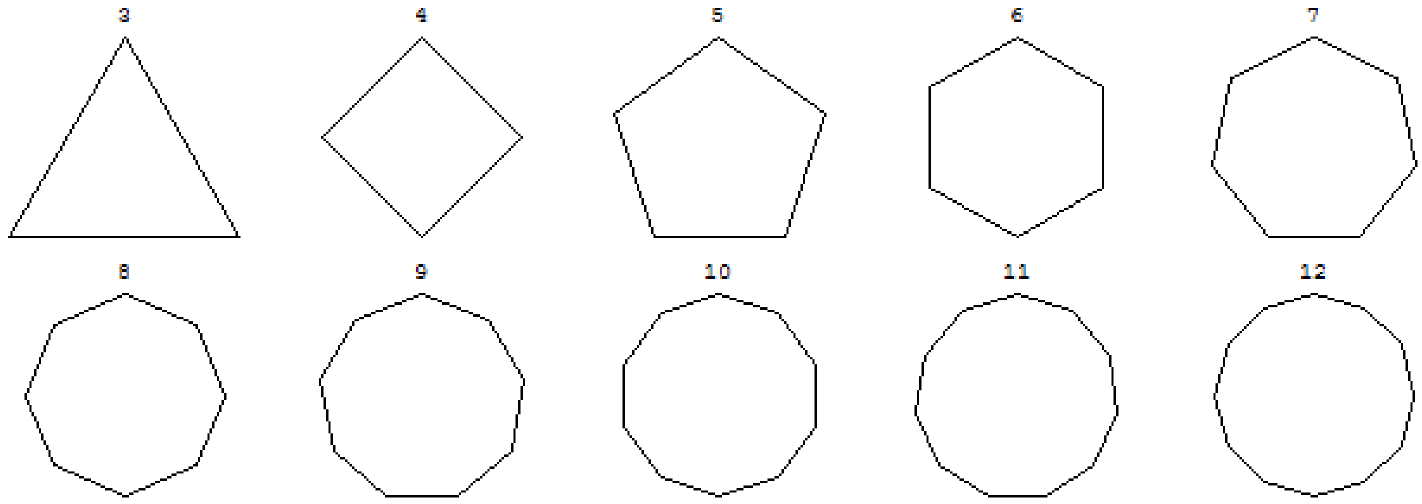
Symmetric structure and its group action

Def.: A set $S \subset \mathbb{R}^3$ is called a symmetric structure if there exists a nontrivial subgroup G of the Euclidean group $E(3)$ under the action of which S is invariant, i.e.:

$$g(S) = g^{-1}(S) = S \quad \forall g \in G. \tag{1}$$

In this case, one says that S has a symmetry group G.

Example: n-gons have symmetry groups given by rotations by a certain angle.



Regular polygons (also called n-gons) for increasing n.

Reconstruction of symmetric objects

Interestingly, the knowledge about symmetry can be a very powerful cue for 3D reconstruction. If we know that the structure looks identical from a different set of view points, then we can actually exploit this information and generate a $3D$ reconstruction – even from a single input image!

Let $g_0 \in SE(3)$ denote the pose of the camera. For all points $\boldsymbol{x}$ of the observed image I we therefore have:

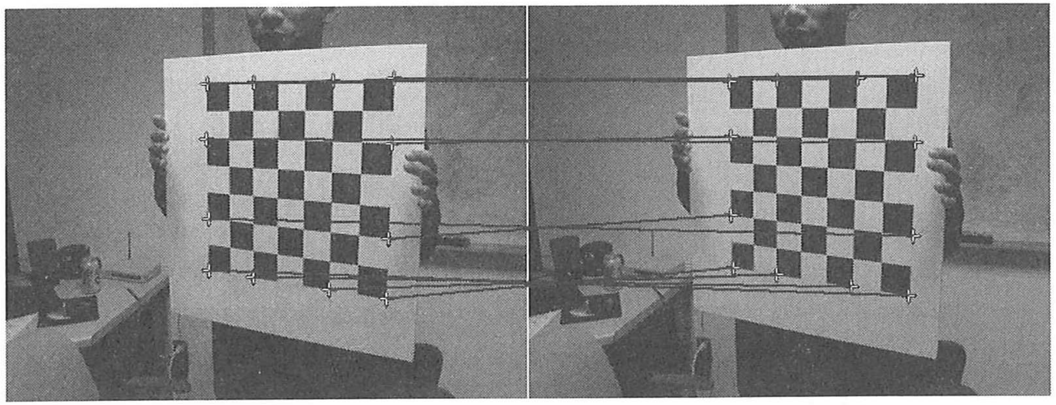
$$\lambda \boldsymbol{x} = \Pi_0 g_o \boldsymbol{X}, \tag{2}$$

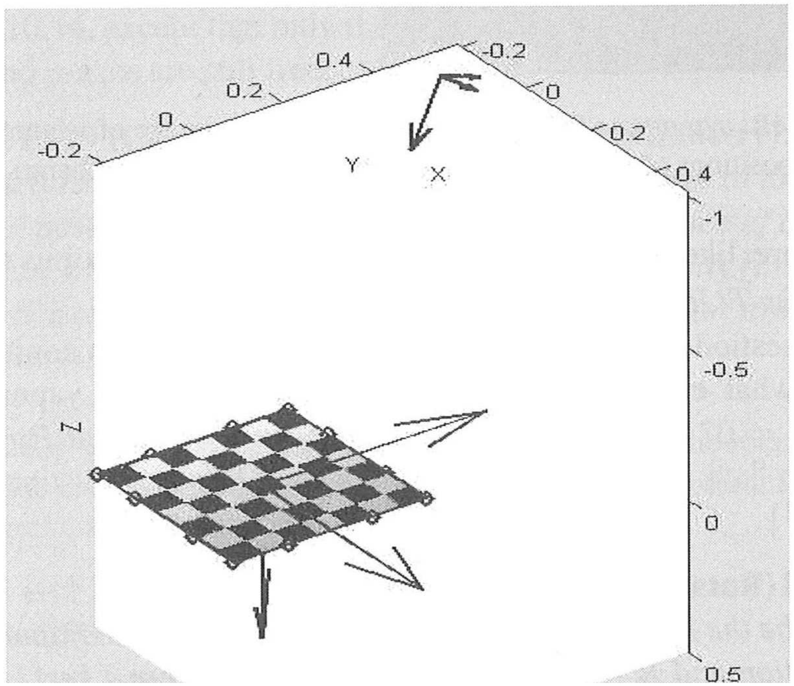
where $\boldsymbol{X}$ are the homogeneous coordinates of a point $p \in S$. Assume that S has the symmetry group G. Then we have:

$$\lambda' \boldsymbol{x}' = \lambda' g(\boldsymbol{x}) = \Pi_0 g_o g \boldsymbol{X} = \Pi_0 (g_0 g g_0^{-1})(g_0 \boldsymbol{X}), \quad \forall g \in G. \tag{3}$$

The second equality states that the structure S remains the same if taken from a vantage point different from the original one by $(g_0 g g_0^{-1})$. As a consequence, we obtain multiple projection equations associated with each point from a single image. In particular, we may in some cases restore the relative pose g_0 from the structure to the viewer.

Reconstruction from a single view





twice the same image with corresponding points pairs.

The symmetric multiple view matrix

If S has a symmetry group G of m elements $g_1, \dots, g_m$, then:

$$\begin{aligned} g_1(\boldsymbol{x}) &\sim \Pi_0 g_0 g_1 g_0^{-1}(g_o \boldsymbol{X}), \\ g_2(\boldsymbol{x}) &\sim \Pi_0 g_0 g_2 g_0^{-1}(g_o \boldsymbol{X}), \\ &\vdots \\ g_m(\boldsymbol{x}) &\sim \Pi_0 g_0 g_m g_0^{-1}(g_o \boldsymbol{X}), \end{aligned} \tag{4}$$

where $\sim$ means equality up to scaling. $g_i' = g_0 g_i g_0^{-1}$ is the relative transformation between original image and the i-th equivalent view. As a consequence, the symmetric multiple view matrix

$$M_s(\boldsymbol{x}) \equiv \begin{pmatrix} \widehat{g_1(\boldsymbol{x})} R_1' \boldsymbol{x} & \widehat{g_1(\boldsymbol{x})} T_1' \\ \widehat{g_2(\boldsymbol{x})} R_2' \boldsymbol{x} & \widehat{g_2(\boldsymbol{x})} T_2' \\ \vdots & \vdots \\ \widehat{g_m(\boldsymbol{x})} R_m' \boldsymbol{x} & \widehat{g_m(\boldsymbol{x})} T_m' \end{pmatrix} \in \mathbb{R}^{3(m-1)\times 2}$$

has $\mathrm{rank}(M_s(\boldsymbol{x})) \leq 1$ for all points $\boldsymbol{x} \in I$.

The symmetric multiple view matrix II

In the symmetric multiple view matrix

$$M_s(\boldsymbol{x}) \equiv \begin{pmatrix} \widehat{g_1(\boldsymbol{x})} R'_1 \boldsymbol{x} & \widehat{g_1(\boldsymbol{x})} T'_1 \\ \widehat{g_2(\boldsymbol{x})} R'_2 \boldsymbol{x} & \widehat{g_2(\boldsymbol{x})} T'_2 \\ \vdots & \vdots \\ \widehat{g_m(\boldsymbol{x})} R'_m \boldsymbol{x} & \widehat{g_m(\boldsymbol{x})} T'_m \end{pmatrix} \in \mathbb{R}^{3(m-1)\times 2}$$

R'_i and T'_i are defined according to $(R'_i, T'_i) = g'_i = g_0 g_i g_0^{-1}$:

$$\begin{cases} R'_i = R_o R_i R_o^\top & \in O(3), \\ T'_i = (I - R_0 R_i R_0^\top) T_0 + R_0 T_i & \in \mathbb{R}^3. \end{cases} \tag{5}$$

The tranformations $g'_i = (R'_i, T'_i)$ can be computed using the multiview factorization approach from the last section. The initial pose g_0 has to be determined from g_i and g'_i according to the condition

$$\boxed{g'_i = g_0 g_i g_0^{-1} \quad \Leftrightarrow \quad g'_i g_0 - g_0 g_i = 0, \quad \forall g_i \in G.}$$

Homographies and symmetry

In the previous lectures we saw that for the case of structures S that are constrained to a plane P in 3D space, the tranformation from P to the image I is given by a homography matrix

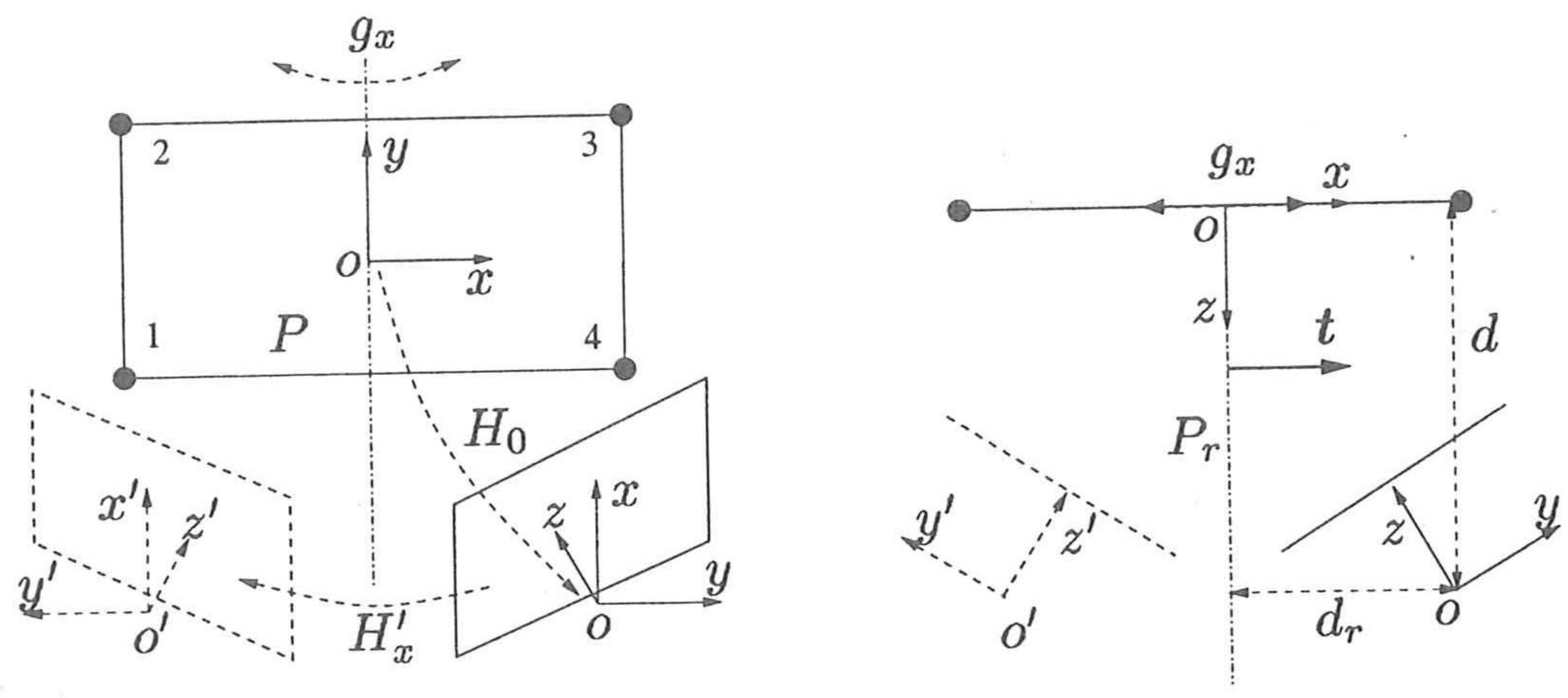
$$H_0 = [R_0(1), R_0(2), T_0] \in \mathbb{R}^{3\times 3},$$

where $R_0(1)$ and $R_0(2)$ are the first two columns of the rotation matrix R_0. If in addition S has the symmetry group G then G is a subgroup of the planar Euclidean group $E(2)$. Elements $g \in G$ can be represented by simple 3×3 matrices (by simply dropping the z-coordinate). Due to symmetry, we have $g(S) = S$ and thus $H_0(g(S)) = H_0(S)$. For a point $\boldsymbol{X} \in S$ we have

$$H_0(g(\boldsymbol{X})) = H_0 g H_0^{-1}(H_0(\boldsymbol{X})).$$

This means that group transformations g acting on the plane P in 3D space correspond to group transformations $H_0 g H_0^{-1}$ acting on the image plane. The conjugate group $G' = H_0 G H_0^{-1}$ is called the homography group. $H' = H_0 g H_0^{-1} \in G'$ represents the homography transformation between two equivalent views.

Homographies and symmetry



Reflection symmetry of a rectangle and the corresponding homography H'.

Homographies and symmetry

We can compute the homography $H' \in G'$ from a single image by associating points that correspond under symmetry. Let $\boldsymbol{x}$ be the (homogeneous) image of a point $\boldsymbol{X}$, and let $\boldsymbol{x}'$ be the image of the symmetric point $g(\boldsymbol{X})$. Then we get:

$$\boldsymbol{x}' = H'\boldsymbol{x} \quad \Leftrightarrow \quad \boldsymbol{x}' \times (H'\boldsymbol{x}) = 0.$$

With 4 such point pairs we can use the 4-point algorithm to determine the homography matrix H' and decompose it according to

$$H' \to \left\{ R', \frac{1}{d}T', N \right\},$$

where N and d are the normal and the offset of the plane. The 3D structure can be obtained by triangulation from two equivalent views. To obtain the original pose, we must determine the homography H_0 such that it fulfills the condition

$$\boxed{H' = H_0 g H_0^{-1} \quad \Leftrightarrow \quad H'H_0 - H_0 g = 0, \quad \forall g \in G}$$

Canonical pose recovery from symmetry

As we saw before, one can extract the homography matrix H' associated with the relative pose between an image and its symmetric counterpart. In order to determine the canonical pose $g_0 = (R_0, T_0)$ defining the location of the structure with respect to the observation point, we need to additionally solve the consistency conditions on the previous slides – so-called Lyapunov equations. It turns out that this can be done uniquely only for certain symmetry groups G.

Proposition 1: Given a (discrete) subgroup $G \in O(3)$, a rotation R_0 is uniquely determined from the pair of sets $(R_0 G R_o^\top, G)$ if and only if the only fixed point of G acting on $\mathbb{R}^3$ is the origin.
Given R_0 it is straight-forward to compute T_0 from the identity

$$T_i' = (I - R_0 R_i R_0^\top) T_0 + R_0 T_i.$$

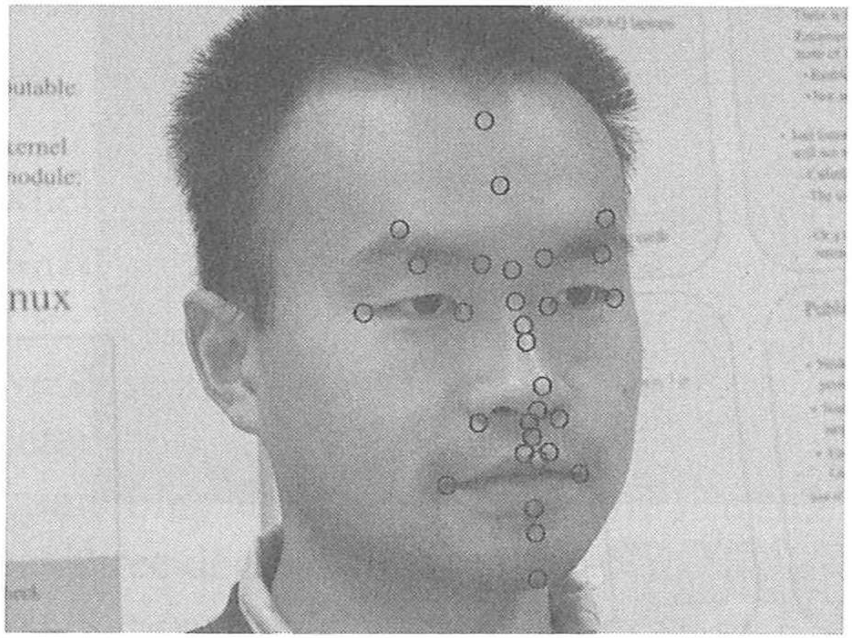
Proposition 2: Suppose that a symmetric strcture S admits a symmetry group G that contains a rotational or reflective subgroup that fixes only the origin of $\mathbb{R}^3$. The the canonical pose g_0 can always be uniquely determined from one image of S.

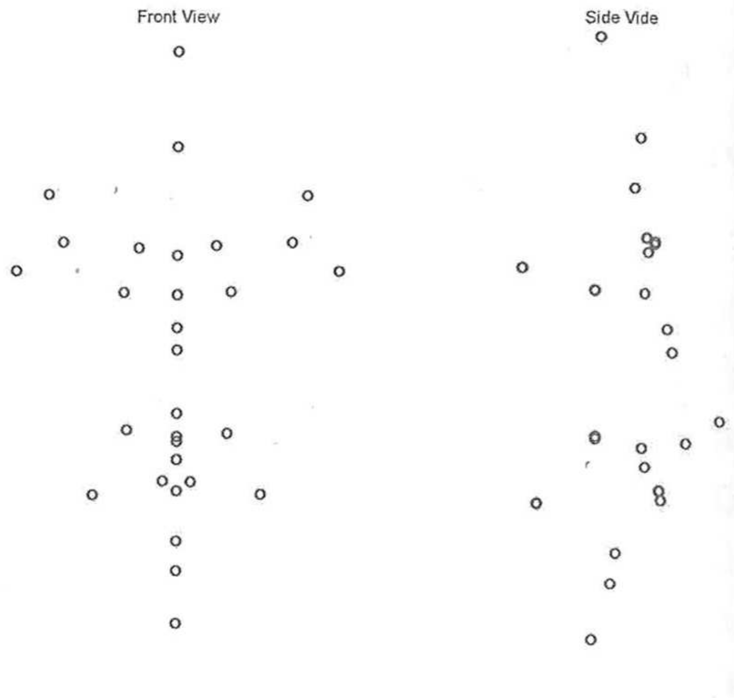
Canonical pose recovery from symmetry

Proposition 3: If a planar symmetric structure S allows a rotational or reflective symmetry subgroup G with two independent rotation or reflection axes, the canonical pose g_0 can always be uniquely determined from one image of S, with the canonical frame origin o restricted in the plane and the z-axis chosen as the plane normal. For a planar symmetric structure, a unique solution for g_0 therefore requires at least two reflections with independent axes, or one reflection and one rotation symmetry.

Proposition 4: Given an image of a structgure S with a reflective symmetry with respect to a plane in $3D$, then the canonical pose g_0 can be determined up to an arbitrary choice of an orthonormal frame in this plane, which is a three-parameter family of ambiguities (i.e. SE(2)). However, if S itself is in a (different) plane, then g_0 is determined up to an arbitrary translation of the frame along the intersection line of the two planes – see slide 11.

Reconstruction from reflective symmetry





Hand-marked symmetric feature points on a human face and 3D position of features.

Source: Ma et al. 2005

Canonical pose from rotational symmetry

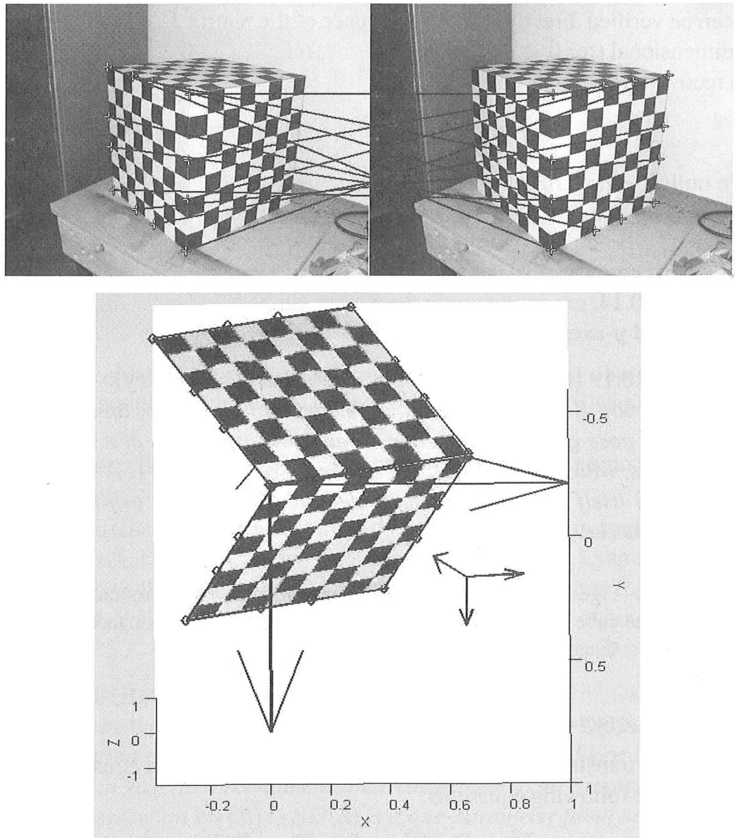
In order to determine the canonical pose g_0, we need to fulfill the consistency conditions. To check whether the solution to this condition is unique, we can define the Lyapunov map

$$L : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}, \quad L(R_0) = R' R_0 - R_0 R,$$

with a the rotation R and $R' = R_0 R R_0^\top$ both known. Analysis of the kernel $ker(L)$, i.e. the set of rotations that are mapped to zero gives us information about the degree of ambiguity in recovering the canonical pose.

Proposition 3: Given an image of a structure S with rotational symmetry with respect to an axis $\omega \in \mathbb{R}^3$, the canonical pose g_0 is determined up to an arbitrary choice of a screw motion along this axis, which is a two-parameter family of ambiguity (i.e. $SO(2) \times \mathbb{R}$). However, if S itself is in a plane, then g_0 is determined up to an arbitrary rotation around the axis ($SO(2)$).

Reconstruction from rotational symmetry



Single view reconstruction of a $3D$ cube

Source: Ma et al. 2005

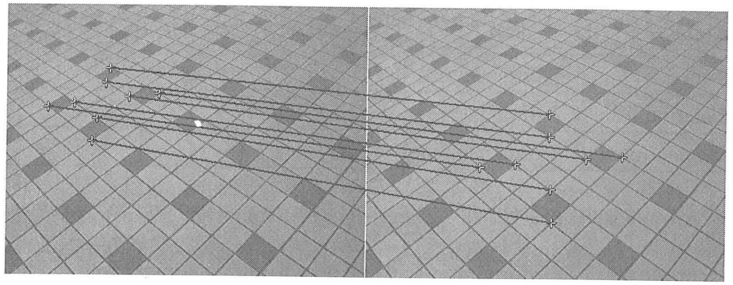
Canonical pose from transl. symmetry

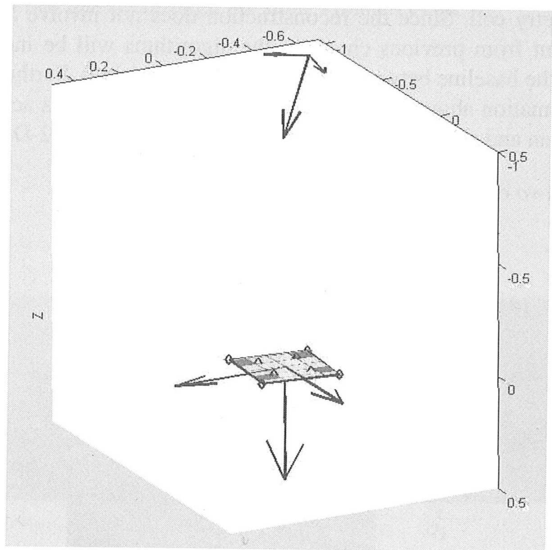
In the case of translational symmetry, we have $R = I$ and $T \neq 0$. Therefore the equations in (5) simplify to

$$R' = R_o I R_0^\top = I, \quad T' = R_0 T.$$

The first equation does not give any information on R_0 (being true for any rotation matrix). The second equation, however allows to infer information on R_0, since both T and T' are known up to a scalar factor). It allows to determine R_0 up to a one-parameter family of rotations, i.e. all rotations $R \in SO(3)$ such that $RT \sim T'$.
If in addition the structure S is planar, one can uniquely recover the $3D$ geometry up to a two-parameter family of ambiguities given by translation of the origin inside the plane.

Reconstruction from transl. symmetry





Single view reconstruction of a planar mosaic floor from translational symmetry

Source: Ma et al. 2005

Reconstruction and the Lyapunov map

The above results can be summarized on the basis of the so-called Lyapunov map, which is defined as

$$L : \mathbb{R}^{4\times4} \to \mathbb{R}^{4\times4}, \quad L(g_0) = g' g_0 - g_0 g.$$

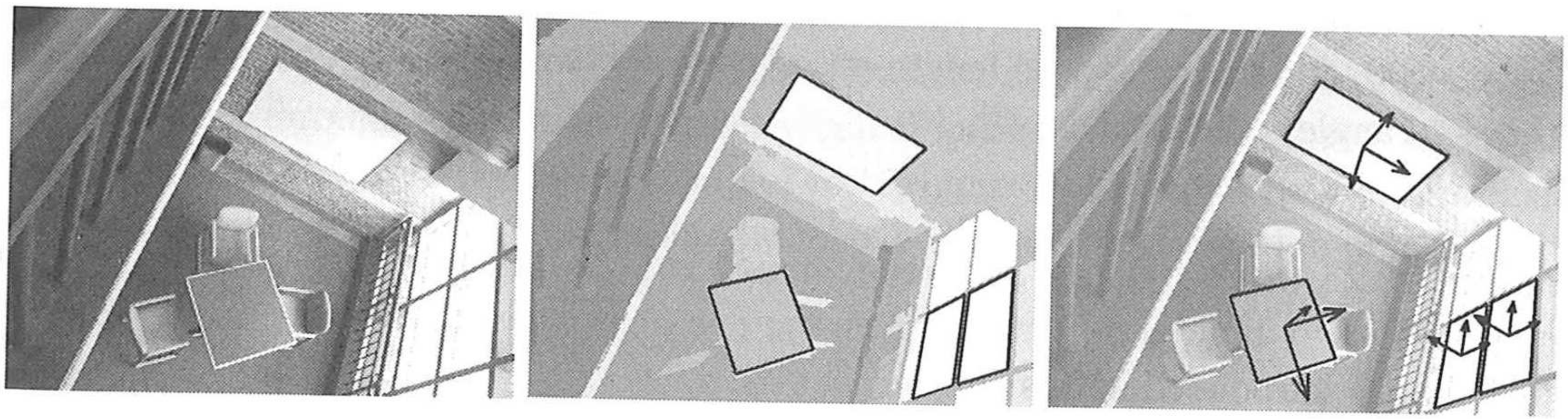
The degree of ambiguity in determining the canonical pose g_0 is related to the size of the kernel $ker(L) = \{g | L(g) = 0\}$.

Specifically we have:

Symmetry	$ker(L)$	g_0 (general scene)	g_0 (planar scene)
Reflective	5-dim.	$SE(2)$	$\mathbb{R}$
Rotational	3-dim.	$SO(2) \times \mathbb{R}$	$SO(2)$
Translational	9-dim.	$SO(2) \times \mathbb{R}^3$	$\mathbb{R}^2$

Multiview reconstruction with symmetry

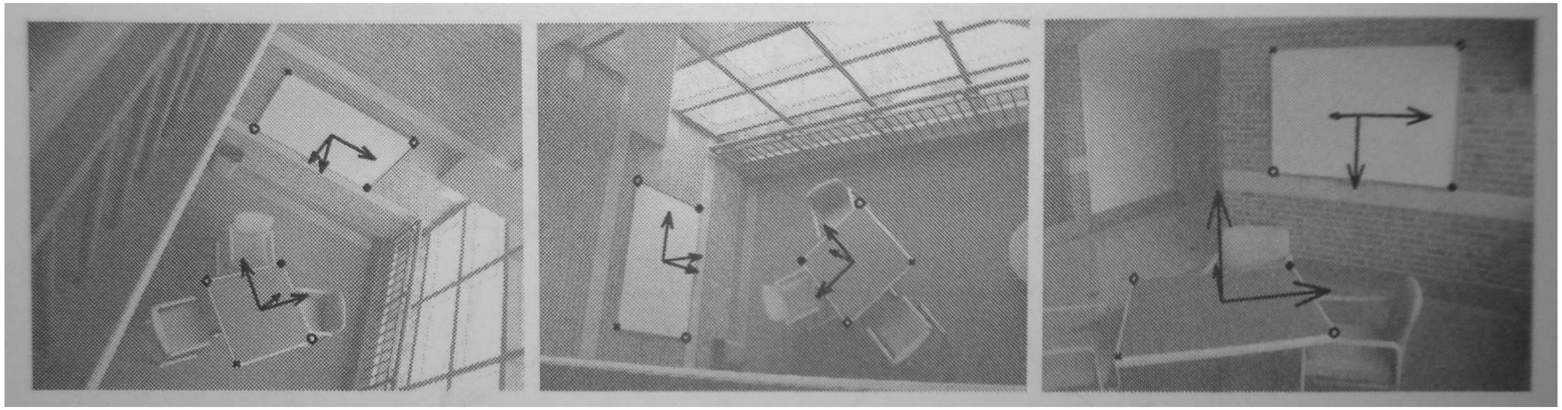
Given multiple views of a symmetric structure, one can tackle the reconstruction as follows: Rather than matching points across images (which is difficult in the large baseline case), one can identify tiles/cells in each image, recover their $3D$ geometry from each image separately and then aim at matching the reconstructed $3D$ structures.



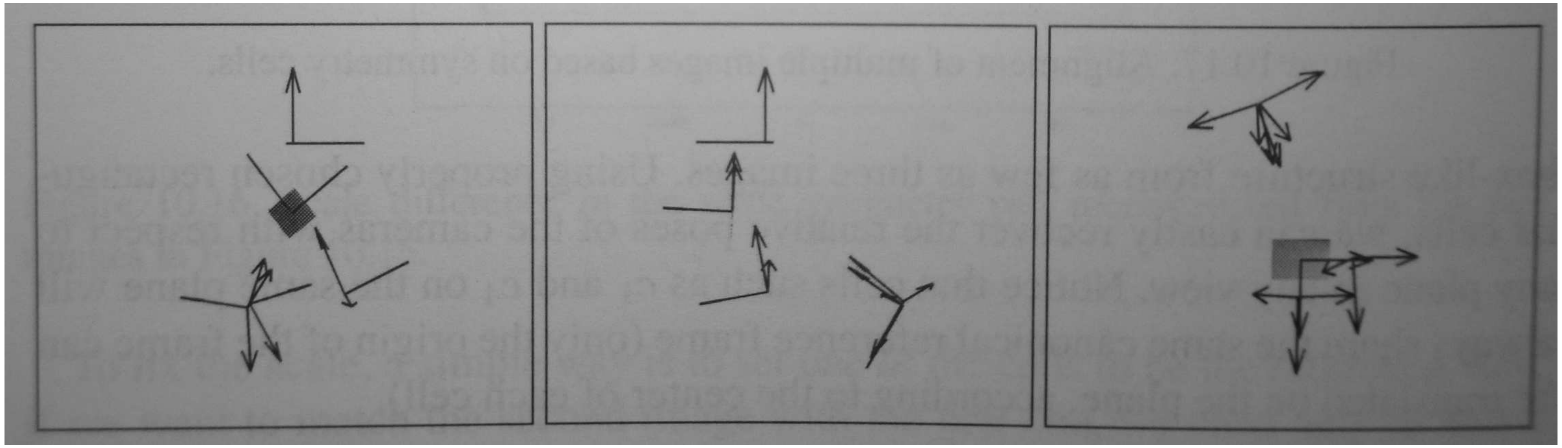
Automatic segmentation and polygon fitting followed by a $3D$ reconstruction of respective cells.

Source: Ma et al. 2005

Multiview reconstruction with symmetry



Automatic matching of respective cells across three images.



Recovered camera poses and cell structures viewed from top, side and front. (Source: Ma et al. 2005)

Summary

We saw that any single image of a symmetric structure is equivalent to multiple equivalent images that are subject to multiview rank conditions or (in the case of planar structures) homography constraints. In general, we only need four points to recover the essential matrix or homography matrix between any two equivalent views. This allows to restore the $3D$ structure and canonical pose from a single image.

There are three fundamental types of isometric 3D symmetry, namely reflection, rotation and translation. We can reconstruct the original pose g_0 uniquely from the Lyapunov equation if we have two independent rotations or reflections in the symmetry group G. Otherwise we can reconstruct up to ambiguities determined by the kernel of the Laypunov map.

Multiview reconstruction of symmetric objects can be done by first reconstructing cells in each individual image and then determining their correspondence across images.

One can also exploit symmetry to retrieve calibration information from images in case of uncalibrated images.