
Multiple View Geometry: Exercise Sheet 3

Solution of the theoretical exercises

1. (a) For all eigenvectors v_i with corresponding eigenvalue λ_i :

$$\begin{aligned}
 \langle v_i, v_i \rangle &= v_i^\top v_i \\
 &= v_i^\top \underbrace{R^\top R}_{=I} v_i \\
 &= \langle Rv_i, Rv_i \rangle \\
 &= \langle \lambda_i v_i, \lambda_i v_i \rangle \\
 &= \bar{\lambda}_i \lambda_i \langle v_i, v_i \rangle \\
 \Leftrightarrow 1 &= \bar{\lambda}_i \lambda_i \quad (\text{because } v_i \neq 0) \\
 \Leftrightarrow 1 &= |\lambda_i|
 \end{aligned}$$

- (b) The characteristic polynomial of a 3×3 -Matrix R is a polynomial of order 3, i.e:

$$\det(R - \lambda Id) = a_3 \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0, \quad \text{with coefficients } a_0, a_1, a_2, a_3 \in \mathbb{R}.$$

$$\begin{aligned}
 &\lambda \text{ is eigenvalue} \\
 \Leftrightarrow &\sum_{k=0}^3 a_k \lambda^k = 0 \\
 \Leftrightarrow &\overline{\sum_{k=0}^3 a_k \lambda^k} = \bar{0} \\
 \Leftrightarrow &\sum_{k=0}^3 \overline{a_k \lambda^k} = 0 \\
 \Leftrightarrow &\sum_{k=0}^3 a_k \bar{\lambda}^k = 0 \\
 \Leftrightarrow &\sum_{k=0}^3 a_k \bar{\lambda}^k = 0 \\
 \Leftrightarrow &\bar{\lambda} \text{ is eigenvalue}
 \end{aligned}$$

(c) $1 = \det(R) = \det(Q\Lambda Q^{-1}) = \det(Q)\det(\Lambda)\det(Q^{-1}) = \det(\Lambda) = \lambda_1 \cdot \lambda_2 \cdot \lambda_3$

(d) Let $\lambda_1 = \cos(\theta) + i \sin(\theta)$ and $\lambda_2 = \cos(\theta) - i \sin(\theta)$.

$$\begin{aligned}
 &\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 1 \\
 \Leftrightarrow &(\cos \theta + i \sin \theta) \cdot (\cos \theta - i \sin \theta) \cdot \lambda_3 = 1 \\
 \Leftrightarrow &(\cos^2 \theta + \sin^2 \theta) \cdot \lambda_3 = 1 \\
 \Leftrightarrow &\lambda_3 = 1
 \end{aligned}$$

The corresponding eigenvector is the rotation axis.

2. $\omega = (\omega_1 \ \omega_2 \ \omega_3)^T$ with $\|\omega\| = 1$, $\hat{\omega} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$

(a)

$$\begin{aligned} \hat{\omega}^2 &= \begin{pmatrix} -(\omega_2^2 + \omega_3^2) & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & -(\omega_1^2 + \omega_3^2) & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & -(\omega_1^2 + \omega_2^2) \end{pmatrix} \\ &= \begin{pmatrix} \underbrace{\omega_1^2 - (\omega_1^2 + \omega_2^2 + \omega_3^2)}_1 & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & \underbrace{\omega_2^2 - (\omega_2^2 + \omega_1^2 + \omega_3^2)}_1 & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & \underbrace{\omega_3^2 - (\omega_3^2 + \omega_1^2 + \omega_2^2)}_1 \end{pmatrix} \\ &= \begin{pmatrix} \omega_1^2 & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & \omega_2^2 & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & \omega_3^2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \omega\omega^\top - \text{Id} \end{aligned}$$

$$\begin{aligned} \hat{\omega}^3 &= \begin{pmatrix} -(\omega_2^2 + \omega_3^2) & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & -(\omega_1^2 + \omega_2^2) & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & -(\omega_1^2 + \omega_2^2) \end{pmatrix} \cdot \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \omega_3 \cdot \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2)}_1 & -\omega_2 \cdot \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2)}_1 \\ -\omega_3 \cdot \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2)}_1 & 0 & \omega_1 \cdot \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2)}_1 \\ \omega_2 \cdot \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2)}_1 & -\omega_1 \cdot \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2)}_1 & 0 \end{pmatrix} \\ &= -\hat{\omega} \end{aligned}$$

(b)

$$\begin{aligned} n \text{ even: } \hat{\omega}^{2n} &= (-1)^{\frac{n}{2}+1} \hat{\omega}^2 \\ n \text{ odd: } \hat{\omega}^{2n} &= (-1)^{\frac{n-1}{2}} \hat{\omega} \end{aligned}$$

Proof via complete induction:

i. for the even numbers n:

A. $n = 2 : \hat{\omega}^2 = (-1)^2 \hat{\omega}^2$

B. Induction step $n \rightarrow n + 2 :$

$$\begin{aligned} \hat{\omega}^{n+2} &= \hat{\omega}^n \cdot \hat{\omega}^2 \\ &\stackrel{(A.)}{=} (-1)^{\frac{n}{2}+1} \cdot \hat{\omega}^2 \cdot \hat{\omega}^2 \\ &= (-1)^{\frac{n}{2}+1} \cdot \hat{\omega}^3 \cdot \hat{\omega} \\ &= (-1)^{\frac{n+2}{2}+1} \cdot \hat{\omega}^2 \end{aligned}$$

ii. for the odd numbers n:

$$A. \ n = 3 : \ \hat{\omega}^3 = -\hat{\omega} = (-1)^{\frac{n-1}{2}} \hat{\omega}$$

B. Induction step $n \rightarrow n + 2$:

$$\begin{aligned} \hat{\omega}^{n+2} &= \hat{\omega}^n \cdot \hat{\omega}^2 \\ &\stackrel{(A.)}{=} (-1)^{\frac{n-1}{2}} \cdot \hat{\omega} \cdot \hat{\omega}^2 \\ &= (-1)^{\frac{n-1}{2}} \cdot \hat{\omega}^3 \\ &= (-1)^{\frac{n-1}{2}+1} \cdot \hat{\omega} \\ &= (-1)^{\frac{n+2-1}{2}} \cdot \hat{\omega} \end{aligned}$$

(c) For $\omega = t\nu$ with $\|\nu\| = 1$ and $t = \|\omega\|$:

$$\begin{aligned} e^{\hat{\omega}} &= e^{\hat{\nu}t} \\ &= \sum_{i=0}^{\infty} \frac{(\hat{\nu})^i}{i!} \\ &= I + \underbrace{\sum_{i=1}^{\infty} (-1)^{i+1} \frac{t^{2i}}{(2i)!} \hat{\nu}^2}_{\sin(t)} + \underbrace{\sum_{i=0}^{\infty} (-1)^i \frac{t^{2i+1}}{(2i+1)!} \hat{\nu}}_{1-\cos(t)} \\ &= I + \sin(t) \frac{\hat{\omega}^2}{\|\omega\|^2} + (1 - \cos(t)) \frac{\hat{\omega}}{\|\omega\|} \end{aligned}$$

3. (a)

$$\dot{g}(t) = \begin{pmatrix} \dot{R}(t) & \dot{T}(t) \\ 0 & 0 \end{pmatrix}$$

$$g(t)^{-1} : \text{If } \begin{pmatrix} y \\ 1 \end{pmatrix} = g(t) \begin{pmatrix} x \\ 1 \end{pmatrix}, \text{ then } R(t)x + T \text{ and}$$

$$x = R(t)^{-1}(y - T) = R(t)^{-1}y - R(t)^{-1}T, \text{ with } R(t)^{-1} = R(t)^T \text{ Therefore}$$

$$g(t)^{-1} = \begin{pmatrix} R(t)^T & -R(t)^T T(t) \\ 0 & 1 \end{pmatrix}$$

(b)

$$\dot{g}(t) \cdot g^{-1}(t) = \begin{pmatrix} \dot{R}(t)R(t)^T & -\dot{R}R(t)^T T(t) + T(t) \\ 0 & 0 \end{pmatrix}$$