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# Multiple View Geometry: Exercise Sheet 8

## Solution of the theoretical exercises

1.

$$\begin{aligned}
 & e_2^\top F = 0 \\
 \Rightarrow & F^\top e_2 = 0 \\
 \Rightarrow & (\hat{T}R)^\top e_2 = 0 \\
 \Rightarrow & R^\top \hat{T}^\top e_2 = 0 \\
 \Rightarrow & R^\top (-T \times e_2) = 0 \quad (\text{because } \hat{T} \text{ is skew-symmetric}) \\
 \Rightarrow & -T \times e_2 = 0 \\
 \Rightarrow & e_2 \sim T
 \end{aligned}$$

$$\begin{aligned}
 & Fe_1 = 0 \\
 \Rightarrow & \hat{T}Re_1 = 0 \\
 \Rightarrow & T \times Re_1 = 0 \\
 \Rightarrow & T \sim Re_1 \\
 \Rightarrow & e_1 \sim R^\top T \\
 \Rightarrow & e_1 \sim R^\top e_2
 \end{aligned}$$

$$2. \exists \lambda \in \mathbb{R} : [R', T'] = \lambda [R, T] * H_p = \lambda [R, T] \begin{bmatrix} I & 0 \\ v^\top & v_4 \end{bmatrix} = \lambda [R + Tv^\top, Tv_4]$$

$$\begin{aligned}
 E' &= \hat{T}'R' \\
 &= (\widehat{\lambda v_4 T}) \cdot (\lambda(R + Tv^\top)) \\
 &= \lambda^2 v_4 \hat{T}(R + Tv^\top) \\
 &= \lambda^2 v_4 \hat{T}R + \lambda^2 v_4 \underbrace{\hat{T}T}_{=0} v^\top \\
 &= \lambda^2 v_4 \hat{T}R \\
 &= \lambda^2 v_4 E \quad \text{with } \lambda^2 v_4 \in \mathbb{R} \\
 \Rightarrow E' &\sim E
 \end{aligned}$$

3.

$$\begin{aligned}
& \text{rank}(N_p) \leq m + 3 \\
\Rightarrow & \text{columns of } N_p \text{ are linearly dependent} \\
\Rightarrow & \exists u \neq 0 \in \mathbb{R}^{m+4} \text{ with } N_p u = 0, u := \begin{bmatrix} X \\ -\lambda \end{bmatrix}, X \in \mathbb{R}^4, -\lambda \in \mathbb{R}^m \\
\Rightarrow & N_p \cdot u = [\Pi, \mathcal{I}] \cdot \begin{bmatrix} X \\ -\lambda \end{bmatrix} = 0 \\
\Rightarrow & \Pi X - \lambda \mathcal{I} = 0 \quad | \cdot \mathcal{I}^\perp \\
\Rightarrow & \mathcal{I}^\perp \Pi X = 0 \\
\Rightarrow & \mathcal{I}^\perp \Pi(\lambda_1 x_1) = 0 \quad (\text{because } \lambda_1 x_1 = \Pi_1 X = [Id, 0]X = X) \\
\Rightarrow & (\hat{x}_t \Pi_t x_1) \cdot \lambda_1 = 0 \quad \forall t = 1, \dots, m \\
\Rightarrow & (\hat{x}_t R_t x_1 + \hat{x}_t T_t) \cdot \binom{\lambda_1}{1} = 0 \quad \forall t = 1, \dots, m \\
\Rightarrow & \begin{pmatrix} \hat{x}_2 R_2 x_1 & \hat{x}_2 T_2 \\ \vdots & \vdots \\ \hat{x}_m R_m x_1 & \hat{x}_m T_m \end{pmatrix} \cdot \binom{\lambda_1}{1} = 0 \\
\Rightarrow & M_p \cdot \binom{\lambda_1}{1} = 0 \\
\Rightarrow & \text{columns of } M_p \text{ are linearly dependent} \\
\Rightarrow & \text{rank}(M_p) \leq 1
\end{aligned}$$

4.  $R_i, T_i$ : relative camera motion between frame 1 and frame  $i$ .

$x_1, \dots, x_m$ : projections of point  $p$  to images  $1, \dots, m$ .

" $\Rightarrow$ ":

$$\begin{aligned}
& \text{rank}(M_p) = 0 \\
\Rightarrow & M_p = 0 \\
\Rightarrow & \begin{pmatrix} \hat{x}_2 R_2 x_1 & \hat{x}_2 T_2 \\ \vdots & \vdots \\ \hat{x}_m R_m x_1 & \hat{x}_m T_m \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \\
\Rightarrow & \hat{x}_i R_i x_1 = 0 \wedge \hat{x}_i T_i = 0 \quad \forall i = 2, \dots, m \\
\Rightarrow & x_i \times R_i x_1 = 0 \wedge x_i \times T_i = 0 \quad \forall i = 2, \dots, m \\
\Rightarrow & x_i \text{ and } R_i x_1 \text{ are linearly dependent, and } x_i \text{ and } T_i \text{ are linearly dependent for all } i = 2, \dots, m. \\
\Rightarrow & x_i \sim R_i x_1 \wedge x_i \sim T_i \quad \forall i = 2, \dots, m \\
\Rightarrow & x_i \sim R_i^\top x_i \wedge x_i \sim T_i \quad \forall i = 2, \dots, m \\
\Rightarrow & x_i \sim R_i^\top e_{i1} \wedge x_i \sim e_{i1} \quad \forall i = 2, \dots, m \quad (\text{because } e_{i1} \sim T_i \text{ for epipole } e_{i1}) \\
\Rightarrow & (i) \quad x_i \sim e_{i1} \quad \forall i = 2, \dots, m \quad (\text{because } e_{i1} \sim R_i^\top e_{i1} \text{ for epipoles } e_{i1} \text{ and } e_{i1}) \\
\Rightarrow & \wedge (ii) \quad x_i \sim e_i \quad \forall i = 2, \dots, m \\
(i) \Rightarrow & \text{All epipoles } e_{12}, \dots, e_{1m} \text{ lie on the same line.} \\
\Rightarrow & \text{All camera centers } o_1, \dots, o_m \text{ lie on the same line.} \\
(ii) \Rightarrow & \text{All images } x_i \text{ lie on the line } \overline{o_1, o_i} \text{ for all } i = 2, \dots, m. \quad (\text{because the epipole lies on that line}) \\
\Rightarrow & \text{The point } p \text{ lies on the line } \overline{o_1, o_i} \text{ for all } i = 2, \dots, m. \\
\Rightarrow & \text{The camera centers } o_1, \dots, o_m \text{ and the point } p \text{ lie on the same line.}
\end{aligned}$$

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” $\Leftarrow$ ”:

$$\begin{aligned} & o_1, \dots, o_m \text{ and } p \text{ lie on the same line.} \\ \Rightarrow & x_1, \dots, x_m \text{ lie on the line.} \\ \Rightarrow & x_i \text{ lies on the line } \overline{o_1, o_i} \text{ for all } i = 1, \dots, m \\ \Rightarrow & x_1 = e_1, \dots, x_m = e_m \text{ for epipoles } e_1, \dots, e_m \\ \Rightarrow & x_i \sim T_i \wedge x_1 \sim R_i^\top x_i \text{ for all } i = 1, \dots, m \\ \Rightarrow & x_i \sim T_i \wedge x_i \sim R_i x_1 \text{ for all } i = 1, \dots, m \\ \Rightarrow & x_i \times T_i = 0 \wedge x_i \times R_i x_1 = 0 \quad \forall i = 1, \dots, m \\ \Rightarrow & \hat{x}_i T_i = 0 \wedge \hat{x}_i R_i x_1 = 0 \text{ for all } i = 1, \dots, m \\ \Rightarrow & \begin{pmatrix} \hat{x}_2 R_2 x_1 & \hat{x}_2 T_2 \\ \vdots & \vdots \\ \hat{x}_m R_m x_1 & \hat{x}_m T_m \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \\ \Rightarrow & M_p = 0 \\ \Rightarrow & \text{rank}(M_p) = 0 \end{aligned}$$

5. The preimages of  $L$  with respect to each camera frame are the following two plane equations:  $l_1^\top X = 0$ , and  $l_2^\top (RX + T) = 0$ . The direction vector  $l$  of the intersecting line  $L$  is perpendicular to both normal vectors of the planes, i.e.  $l = l_1 \times R^\top l_2$ . The position vector  $p$  can be computed by solving the linear equation system  $l_1^\top p = 0 \wedge l_2^\top Rp + l_2^\top T = 0$ .