



# Multiple View Geometry: Solution Exercise Sheet 2

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<http://vision.in.tum.de/teaching/ss2013/mvg2013>

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## Part I: Theory

$$1. \quad (a) \quad M = \begin{pmatrix} I & T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(b) \quad M = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(c) \quad M = \begin{pmatrix} I & T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(d) \quad M = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I & T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R & RT \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & r_1 t_x \\ r_{21} & r_{22} & r_{23} & r_2 t_y \\ r_{31} & r_{32} & r_{33} & r_3 t_z \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where  $r_1, r_2, r_3$  are the row vectors of  $R$ :  $R = \begin{pmatrix} -r_1- \\ -r_2- \\ -r_3- \end{pmatrix}$ .

$$2. \quad " \Rightarrow " : \quad x^\top M_1 x = x^\top M_2 x \quad \text{for all } x \in \mathbb{R}^3$$

$$\Leftrightarrow x^\top M_1 x - x^\top M_2 x = 0$$

$$\Leftrightarrow x^\top (M_1 - M_2)x = 0$$

$$\Leftrightarrow (x_1 \ x_2 \ x_3) \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0, \text{ where } M_1 - M_2 := \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$$

$$\begin{aligned} \Leftrightarrow & m_{11}x_1^2 + m_{12}x_1x_2 + m_{13}x_1x_3 \\ & + m_{22}x_2^2 + m_{21}x_1x_2 + m_{23}x_2x_3 \\ & + m_{33}x_3^2 + m_{31}x_1x_3 + m_{32}x_2x_3 = 0 \end{aligned}$$

$$\Leftrightarrow m_{11}x_1^2 + m_{22}x_2^2 + m_{33}x_3^2 + (m_{12} + m_{21})x_1x_2 + (m_{13} + m_{31})x_1x_3 + (m_{23} + m_{32})x_2x_3 = 0$$

$$\Rightarrow m_{11} = 0 \wedge m_{22} = 0 \wedge m_{33} = 0 \wedge m_{12} = -m_{21} \wedge m_{13} = -m_{31} \wedge m_{23} = -m_{32}$$

$$\Rightarrow M_1 - M_2 \in so(3).$$

$$\Rightarrow M_1 - M_2 \in so(3) \Rightarrow M_1 - M_2 := \begin{pmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{pmatrix}$$

Let  $x \in \mathbb{R}^3$

$$\begin{aligned} \Rightarrow x^\top (M_1 - M_2)x &= (x_1 \ x_2 \ x_3) \begin{pmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ &= ax_1x_2 - bx_1x_3 - ax_2x_1 + cx_3x_2 + bx_1x_3 - cx_2x_3 = 0 \\ \Rightarrow x^\top M_1 x &= x^\top M_2 x \quad \text{for all } x \in \mathbb{R}^3 \end{aligned}$$

Another possible solution:

$$\begin{aligned} \Rightarrow x^\top M_1 x &= x^\top M_2 x \quad \text{for all } x \in \mathbb{R}^3 \\ \Leftrightarrow \langle x, M_1 x \rangle &= \langle x, M_2 x \rangle \quad \text{for all } x \in \mathbb{R}^3 \\ \Leftrightarrow \langle x, (M_1 - M_2)x \rangle &= 0 \quad \text{for all } x \in \mathbb{R}^3 \\ \Rightarrow \text{in particular: a) } \langle e_i, (M_1 - M_2)e_i \rangle &= 0 \quad \text{where } e_i = \text{i-th unit vector} \\ \text{and b) } \langle e_i + e_j, (M_1 - M_2)(e_i + e_j) \rangle &= 0 \quad \text{where } e_j = \text{j-th unit vector} \end{aligned}$$

$$\text{Let } (M_1 - M_2) = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}.$$

$$\begin{aligned} \text{a) } \Rightarrow m_{ii} &= 0 \\ \text{b) } \Rightarrow 0 &= m_{ii} + m_{ij} + m_{ji} + m_{jj} = m_{ij} + m_{ji} \Rightarrow m_{ij} = -m_{ji} \\ \Rightarrow M_1 - M_2 &\in so(3). \end{aligned}$$

$$\Rightarrow \text{We assume: } M_1 - M_2 \in so(3) \Rightarrow (M_1 - M_2)^\top = -(M_1 - M_2).$$

$$\begin{aligned} \langle (M_1 - M_2)x, x \rangle &= \langle x, (M_1 - M_2)x \rangle \quad \text{for all } x \in \mathbb{R}^3 \\ \Leftrightarrow x^\top (M_1 - M_2)^\top x &= x^\top (M_1 - M_2)x \\ \Leftrightarrow -x^\top (M_1 - M_2)x &= x^\top (M_1 - M_2)x \quad (\text{assuming } M_1 - M_2 \in so(3)) \\ \Leftrightarrow -x^\top M_1 x + x^\top M_2 x &= x^\top M_1 x - x^\top M_2 x \\ \Leftrightarrow 2x^\top M_1 x &= 2x^\top M_2 x \\ \Leftrightarrow x^\top M_1 x &= x^\top M_2 x \quad \text{for all } x \in \mathbb{R}^3 \end{aligned}$$

3. We show that:  $x \in \ker(A) \Leftrightarrow x \in \ker(A^\top A)$ .

$$\begin{aligned} \Rightarrow & \text{ Let } x \in \ker(A) \\ & \Rightarrow Ax = 0 \Rightarrow A^\top Ax = 0 \Rightarrow x \in \ker(A^\top A) \end{aligned}$$

$$\begin{aligned} \Rightarrow & \text{ Let } x \in \ker(A^\top A) \text{ and } A = \begin{pmatrix} | & | \\ a_1 & \cdots & a_n \\ | & | \end{pmatrix} \in \mathbb{R}^{m \times n} \end{aligned}$$

$$\begin{aligned} \Rightarrow & 0 = A^\top Ax = \begin{pmatrix} -a_1^\top & - \\ \vdots & \\ -a_n^\top & - \end{pmatrix} \begin{pmatrix} | & | \\ a_1 & \cdots & a_n \\ | & | \end{pmatrix} x = \begin{pmatrix} a_1^\top a_1 & \cdots & a_1^\top a_n \\ \vdots & & \vdots \\ a_n^\top a_1 & \cdots & a_n^\top a_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \sum_i a_1^\top a_i x_i \\ \vdots \\ \sum_i a_n^\top a_i x_i \end{pmatrix} \end{aligned}$$

$$\Rightarrow a_1^\top \sum_i a_i x_i = 0 \wedge \dots \wedge a_n^\top \sum_i a_i x_i = 0$$

$$\Leftrightarrow a_j^\top \sum_i a_i x_i = 0 \text{ for all } j = 1, \dots, n$$

$$\Rightarrow a) a_j = 0 \text{ for all } j = 1, \dots, n \Rightarrow A = 0 \Rightarrow Ax = 0 \Rightarrow x \in \ker(A)$$

$$\Rightarrow b) \sum_i a_i x_i = 0 \Leftrightarrow Ax = 0 \Leftrightarrow x \in \ker(A)$$

Another possible solution:

$$\begin{aligned} \Leftarrow & \text{ Let } x \in \ker(A^\top A) \\ & \Rightarrow A^\top Ax = 0 \\ & \Rightarrow 0 = \langle y, 0 \rangle = \langle y, A^\top Ax \rangle = \langle Ay, Ax \rangle \text{ for all } y \in \mathbb{R}^n \text{ (and in particular for } y = x) \\ & \Rightarrow 0 = \langle Ax, Ax \rangle = \|Ax\|^2 \Rightarrow Ax = 0 \Leftrightarrow x \in \ker(A) \end{aligned}$$