Multiple View Geometry: Solution Exercise Sheet 3

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Part I: Theory

1. (a) $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ eigenvalues of $R \quad \Rightarrow \quad \exists\, v_1, v_2, v_3 \neq 0 : Rv_j = \lambda_j v_j \quad \forall j$

Let v_j be eigenvector of R, $\lambda_j = a + ib \in \mathbb{C}$ the corresponding eigenvalue:

$$\begin{aligned} < v_j, v_j > &= v_j^\top v_j \\ &= v_j^\top R^\top R v_j \qquad (R \in SO(3) \Rightarrow R^\top R = I) \\ &= < Rv_j, Rv_j > \\ &= < \lambda_j v_j, \lambda_j v_j > \\ &= \lambda_j \bar{\lambda}_j < v_j, v_j > \\ \Leftrightarrow \quad 1 &= \lambda_j \bar{\lambda}_j = |\lambda_j|^2 \qquad (\text{because } v_j \neq 0) \\ \Rightarrow \quad 1 &= |\lambda_j| \qquad (|\lambda_j| = \sqrt{a^2 + b^2} \geq 0 \ \text{ as } \ a, b \in \mathbb{R}) \end{aligned}$$

(b) To show:
If $\exists\, v \neq 0 : Rv = \lambda v \quad \Rightarrow \quad \exists\, w \neq 0 : Rw = \bar{\lambda} w$

$$\begin{aligned} \lambda \in \mathbb{C}, R \in \mathbb{R}^{3\times3} : \quad Rv = \lambda v \quad &\Leftrightarrow \quad \overline{Rv} = \overline{\lambda v} \\ &\Leftrightarrow \quad R\bar{v} = \bar{\lambda}\bar{v} \\ &\Leftrightarrow \quad \bar{\lambda} \text{ eigenvalue to } R\text{, with } \bar{v} \text{ as an eigenvector.} \end{aligned}$$

(c) $1 \overset{R \in SO(3)}{=} \det(R) \overset{\text{EVD}}{=} \det(Q\Lambda Q^{-1}) = \det(Q)\det(\Lambda)\det(Q^{-1}) = \det(\Lambda) = \lambda_1 \cdot \lambda_2 \cdot \lambda_3$

(d) To show: $\exists\, j : \lambda_j = 1$

$$\begin{aligned} 1 &\overset{(c)}{=} \lambda_1 \lambda_2 \lambda_3 \\ &\overset{(b)}{=} \lambda_1 \lambda_2 \overline{\lambda_2} \qquad (\text{Wlog we assume } \lambda_3 = \overline{\lambda_2}. \quad \text{E.g. } \lambda_2 = \cos\theta + i \sin\theta) \\ &= \lambda_1 |\lambda_2|^2 \qquad (\text{In (a) we saw: } |\lambda_j| = 1 \quad \forall\, j) \\ &\overset{(a)}{=} \lambda_1 \end{aligned}$$

The corresponding eigenvector is the rotation axis. ($Rv = v$)

2. We know: $\omega = (\omega_1\ \omega_2\ \omega_3)^T$ with $||\omega|| = 1$ and $\hat{\omega} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$.

(a)

$$
\begin{aligned}
\hat{\omega}^2 &= \begin{pmatrix} -(\omega_2^2+\omega_3^2) & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & -(\omega_1^2+\omega_3^2) & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & -(\omega_1^2+\omega_2^2) \end{pmatrix} \\
&= \begin{pmatrix} \omega_1^2 - \underbrace{(\omega_1^2+\omega_2^2+\omega_3^2)}_{1} & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & \omega_2^2 - \underbrace{(\omega_2^2+\omega_1^2+\omega_3^2)}_{1} & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & \omega_3^2 - \underbrace{(\omega_3^2+\omega_1^2+\omega_2^2)}_{1} \end{pmatrix} \\
&= \begin{pmatrix} \omega_1^2 & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & \omega_2^2 & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & \omega_3^2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \omega\omega^\top - \mathbf{I}
\end{aligned}
$$

$$
\begin{aligned}
\hat{\omega}^3 &= \begin{pmatrix} -(\omega_2^2+\omega_3^2) & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & -(\omega_1^2+\omega_2^2) & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & -(\omega_1^2+\omega_2^2) \end{pmatrix} \cdot \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \omega_3 \cdot \underbrace{(\omega_1^2+\omega_2^2+\omega_3^2)}_{1} & -\omega_2 \cdot \underbrace{(\omega_1^2+\omega_2^2+\omega_3^2)}_{1} \\ -\omega_3 \cdot \underbrace{(\omega_1^2+\omega_2^2+\omega_3^2)}_{1} & 0 & \omega_1 \cdot \underbrace{(\omega_1^2+\omega_2^2+\omega_3^2)}_{1} \\ \omega_2 \cdot \underbrace{(\omega_1^2+\omega_2^2+\omega_3^2)}_{1} & -\omega_1 \cdot \underbrace{(\omega_1^2+\omega_2^2+\omega_3^2)}_{1} & 0 \end{pmatrix} \\
&= -\hat{\omega}
\end{aligned}
$$

(b) n even: $\hat{\omega}^n = (-1)^{\frac{n}{2}+1}\,\hat{\omega}^2$

n odd: $\hat{\omega}^n = (-1)^{\frac{n-1}{2}}\,\hat{\omega}$

Proof via complete induction:

i. For even numbers n:

- $n = 2:\quad \hat{\omega}^2 = (-1)^{\frac{2}{2}+1}\hat{\omega}^2$
- Induction step $n \to n+2$:

$$
\begin{aligned}
\hat{\omega}^{n+2} &= \hat{\omega}^n \cdot \hat{\omega}^2 \\
&= (-1)^{\frac{n}{2}+1} \cdot \hat{\omega}^2 \cdot \hat{\omega}^2 \qquad \text{(assumption)} \\
&= (-1)^{\frac{n}{2}+1} \cdot \hat{\omega}^3 \cdot \hat{\omega} \\
&\overset{(a)}{=} (-1)^{\frac{(n+2)}{2}+1} \cdot \hat{\omega}^2
\end{aligned}
$$

ii. For odd numbers n:
   - $n = 3: \quad \hat{\omega}^3 = -\hat{\omega} = (-1)^{\frac{3-1}{2}} \hat{\omega}$
   - Induction step $n \to n+2$:

$$\begin{aligned}
\hat{\omega}^{n+2} &= \hat{\omega}^n \cdot \hat{\omega}^2 \\
&= (-1)^{\frac{n-1}{2}} \cdot \hat{\omega} \cdot \hat{\omega}^2 \qquad \text{(assumption)} \\
&= (-1)^{\frac{n-1}{2}} \cdot \hat{\omega}^3 \\
&\overset{(a)}{=} (-1)^{\frac{n-1}{2}+1} \cdot \hat{\omega} \\
&= (-1)^{\frac{(n+2)-1}{2}} \cdot \hat{\omega}
\end{aligned}$$

(c) We know: $\omega \in \mathbb{R}^3$. Let $v = \frac{\omega}{\|\omega\|}$ and $t = \|\omega\|$. Hence, $w = vt$.

$$\begin{aligned}
e^{\hat{\omega}} &= e^{\hat{\nu} t} \\
&= \sum_{n=0}^{\infty} \frac{(\hat{\nu} t)^n}{n!} \\
&\overset{(b)}{=} I + \underbrace{\sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^{2n}}{(2n)!}}_{1-\cos(t)} \hat{\nu}^2 + \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!}}_{\sin(t)} \hat{\nu} \\
&\overset{\text{(def.)}}{=} I + \frac{\hat{\omega}^2}{\|\omega\|^2} (1 - \cos(\|\omega\|)) + \frac{\hat{\omega}}{\|\omega\|} \sin(\|\omega\|)
\end{aligned}$$

3. (a) $\dot{g}(t) = \begin{pmatrix} \dot{R}(t) & \dot{T}(t) \\ 0 & 0 \end{pmatrix}$

$$g(t)^{-1} = \begin{pmatrix} R(t)^{-1} & -R(t)^{-1} T(t) \\ 0 & 1 \end{pmatrix} \overset{R \in SO(3)}{=} \begin{pmatrix} R(t)^\top & -R(t)^\top T(t) \\ 0 & 1 \end{pmatrix}$$

(b) $\dot{g}(t) \cdot g^{-1}(t) = \begin{pmatrix} \dot{R}(t) R(t)^\top & -\dot{R} R(t)^\top T(t) + \dot{T}(t) \\ 0 & 0 \end{pmatrix}$