



# Multiple View Geometry: Solution Exercise Sheet 3

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## Part I: Theory

1. (a)  $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$  eigenvalues of  $R \Rightarrow \exists v_1, v_2, v_3 \neq 0 : Rv_j = \lambda_j v_j \quad \forall j$

Let  $v_j$  be eigenvector of  $R$ ,  $\lambda_j = a + ib \in \mathbb{C}$  the corresponding eigenvalue:

$$\begin{aligned} \langle v_j, v_j \rangle &= v_j^\top v_j \\ &= v_j^\top R^\top R v_j \quad (R \in SO(3) \Rightarrow R^\top R = I) \\ &= \langle Rv_j, Rv_j \rangle \\ &= \langle \lambda_j v_j, \lambda_j v_j \rangle \\ &= \lambda_j \bar{\lambda}_j \langle v_j, v_j \rangle \\ \Leftrightarrow 1 &= \lambda_j \bar{\lambda}_j = |\lambda_j|^2 \quad (\text{because } v_j \neq 0) \\ \Rightarrow 1 &= |\lambda_j| \quad (|\lambda_j| = \sqrt{a^2 + b^2} \geq 0 \text{ as } a, b \in \mathbb{R}) \end{aligned}$$

- (b) To show:

$$\text{If } \exists v \neq 0 : Rv = \lambda v \Rightarrow \exists w \neq 0 : Rw = \bar{\lambda} w$$

$$\lambda \in \mathbb{C}, R \in \mathbb{R}^{3 \times 3} : Rv = \lambda v \Leftrightarrow \overline{Rv} = \overline{\lambda v}$$

$$\Leftrightarrow R\bar{v} = \bar{\lambda} \bar{v}$$

$$\Leftrightarrow \bar{\lambda} \text{ eigenvalue to } R, \text{ with } \bar{v} \text{ as an eigenvector.}$$

(c)  $1 \stackrel{R \in SO(3)}{=} \det(R) \stackrel{\text{EVD}}{=} \det(Q\Lambda Q^{-1}) = \det(Q)\det(\Lambda)\det(Q^{-1}) = \det(\Lambda) = \lambda_1 \cdot \lambda_2 \cdot \lambda_3$

- (d) To show:  $\exists j : \lambda_j = 1$

$$1 \stackrel{(c)}{=} \lambda_1 \lambda_2 \lambda_3$$

$$\stackrel{(b)}{=} \lambda_1 \lambda_2 \bar{\lambda}_2 \quad (\text{Wlog we assume } \lambda_3 = \bar{\lambda}_2. \text{ E.g. } \lambda_2 = \cos \theta + i \sin \theta)$$

$$= \lambda_1 |\lambda_2|^2 \quad (\text{In (a) we saw: } |\lambda_j| = 1 \quad \forall j)$$

$$\stackrel{(a)}{=} \lambda_1$$

The corresponding eigenvector is the rotation axis. ( $Rv = v$ )

2. We know:  $\omega = (\omega_1 \ \omega_2 \ \omega_3)^T$  with  $\|\omega\| = 1$  and  $\hat{\omega} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$ .

(a)

$$\begin{aligned} \hat{\omega}^2 &= \begin{pmatrix} -(\omega_2^2 + \omega_3^2) & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & -(\omega_1^2 + \omega_3^2) & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & -(\omega_1^2 + \omega_2^2) \end{pmatrix} \\ &= \begin{pmatrix} \omega_1^2 - \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2)}_1 & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & \omega_2^2 - \underbrace{(\omega_2^2 + \omega_1^2 + \omega_3^2)}_1 & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & \omega_3^2 - \underbrace{(\omega_3^2 + \omega_1^2 + \omega_2^2)}_1 \end{pmatrix} \\ &= \begin{pmatrix} \omega_1^2 & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & \omega_2^2 & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & \omega_3^2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \omega\omega^T - \mathbf{I} \end{aligned}$$

$$\begin{aligned} \hat{\omega}^3 &= \begin{pmatrix} -(\omega_2^2 + \omega_3^2) & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & -(\omega_1^2 + \omega_2^2) & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & -(\omega_1^2 + \omega_2^2) \end{pmatrix} \cdot \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \omega_3 \cdot \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2)}_1 & -\omega_2 \cdot \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2)}_1 \\ -\omega_3 \cdot \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2)}_1 & 0 & \omega_1 \cdot \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2)}_1 \\ \omega_2 \cdot \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2)}_1 & -\omega_1 \cdot \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2)}_1 & 0 \end{pmatrix} \\ &= -\hat{\omega} \end{aligned}$$

(b)  $n$  even:  $\hat{\omega}^n = (-1)^{\frac{n}{2}+1} \hat{\omega}^2$   
 $n$  odd:  $\hat{\omega}^n = (-1)^{\frac{n-1}{2}} \hat{\omega}$

Proof via complete induction:

i. For even numbers  $n$ :

- $n = 2$ :  $\hat{\omega}^2 = (-1)^{\frac{2}{2}+1} \hat{\omega}^2$
- Induction step  $n \rightarrow n + 2$ :

$$\begin{aligned} \hat{\omega}^{n+2} &= \hat{\omega}^n \cdot \hat{\omega}^2 \\ &= (-1)^{\frac{n}{2}+1} \cdot \hat{\omega}^2 \cdot \hat{\omega}^2 \quad (\text{assumption}) \\ &= (-1)^{\frac{n}{2}+1} \cdot \hat{\omega}^3 \cdot \hat{\omega} \\ &\stackrel{(a)}{=} (-1)^{\frac{(n+2)}{2}+1} \cdot \hat{\omega}^2 \end{aligned}$$

ii. For odd numbers  $n$ :

-  $n = 3$  :  $\hat{\omega}^3 = -\hat{\omega} = (-1)^{\frac{3-1}{2}} \hat{\omega}$

- Induction step  $n \rightarrow n + 2$  :

$$\begin{aligned} \hat{\omega}^{n+2} &= \hat{\omega}^n \cdot \hat{\omega}^2 \\ &= (-1)^{\frac{n-1}{2}} \cdot \hat{\omega} \cdot \hat{\omega}^2 && \text{(assumption)} \\ &= (-1)^{\frac{n-1}{2}} \cdot \hat{\omega}^3 \\ &\stackrel{(a)}{=} (-1)^{\frac{n-1}{2}+1} \cdot \hat{\omega} \\ &= (-1)^{\frac{(n+2)-1}{2}} \cdot \hat{\omega} \end{aligned}$$

(c) We know:  $\omega \in \mathbb{R}^3$ . Let  $v = \frac{\omega}{\|\omega\|}$  and  $t = \|\omega\|$ . Hence,  $w = vt$ .

$$\begin{aligned} e^{\hat{\omega}} &= e^{\hat{v}t} \\ &= \sum_{n=0}^{\infty} \frac{(\hat{v}t)^n}{n!} \\ &\stackrel{(b)}{=} I + \underbrace{\sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^{2n}}{(2n)!}}_{1 - \cos(t)} \hat{v}^2 + \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!}}_{\sin(t)} \hat{v} \\ &\stackrel{(\text{def.})}{=} I + \frac{\hat{\omega}^2}{\|\omega\|^2} (1 - \cos(\|\omega\|)) + \frac{\hat{\omega}}{\|\omega\|} \sin(\|\omega\|) \end{aligned}$$

3. (a)  $\dot{g}(t) = \begin{pmatrix} \dot{R}(t) & \dot{T}(t) \\ 0 & 0 \end{pmatrix}$

$$g(t)^{-1} = \begin{pmatrix} R(t)^{-1} & -R(t)^{-1}T(t) \\ 0 & 1 \end{pmatrix} \stackrel{R \in SO(3)}{=} \begin{pmatrix} R(t)^\top & -R(t)^\top T(t) \\ 0 & 1 \end{pmatrix}$$

(b)  $\dot{g}(t) \cdot g^{-1}(t) = \begin{pmatrix} \dot{R}(t)R(t)^\top & -\dot{R}R(t)^\top T(t) + T(t) \\ 0 & 0 \end{pmatrix}$