Analysis of Three-Dimensional Shapes
(IN2238, TU München, Summer 2014)

Shapes as Metric Spaces
(14.04.2014)

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Announcement

There will be no class on 21.04.2013

(further announcements via e-mail)

Seminar

"Laplace-Beltrami Operator"

Emanuel Laude
Frank Schmidt

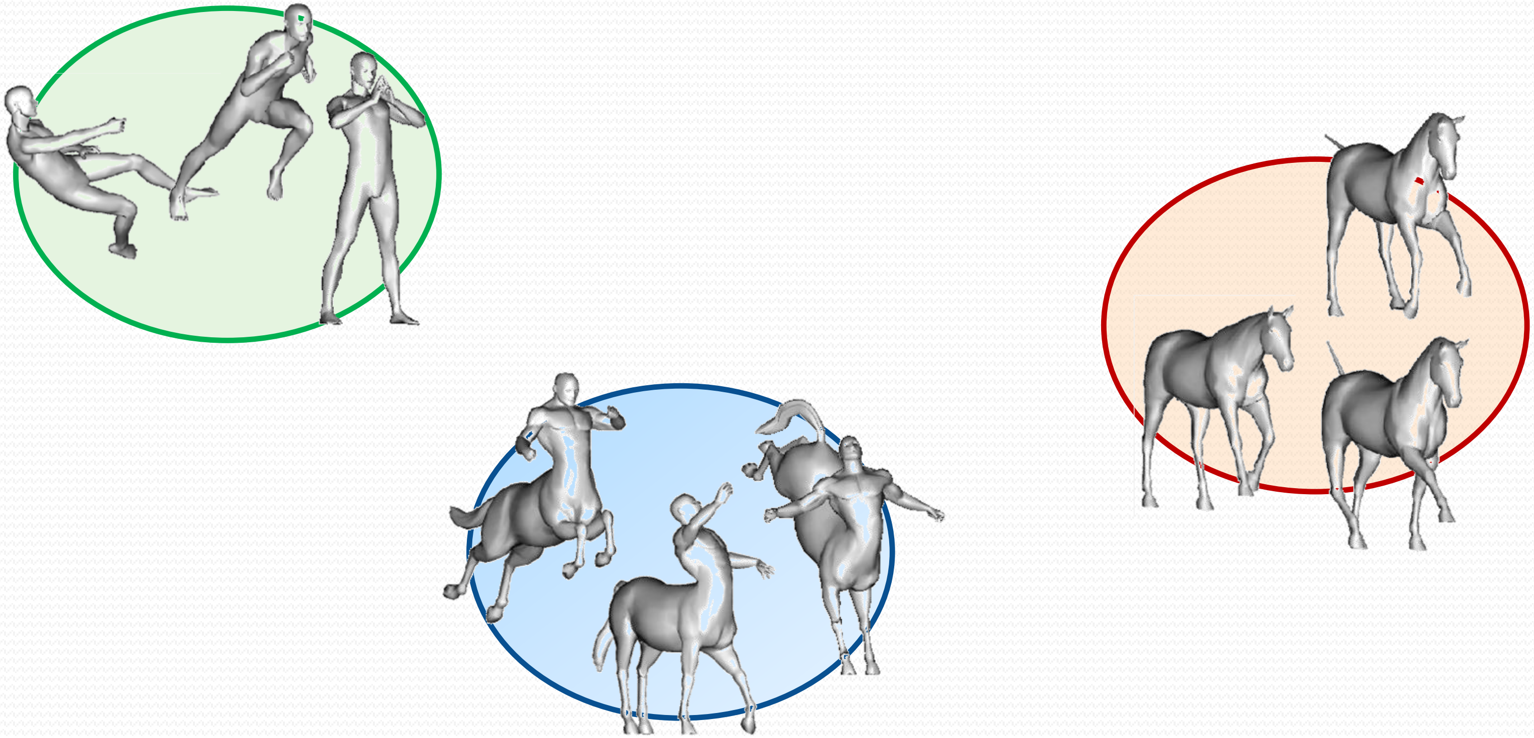
Wednesday, April 16
14:00 Room 02.09.023

Seminar

"Discrete differential geometry"
Thorsten Philipp

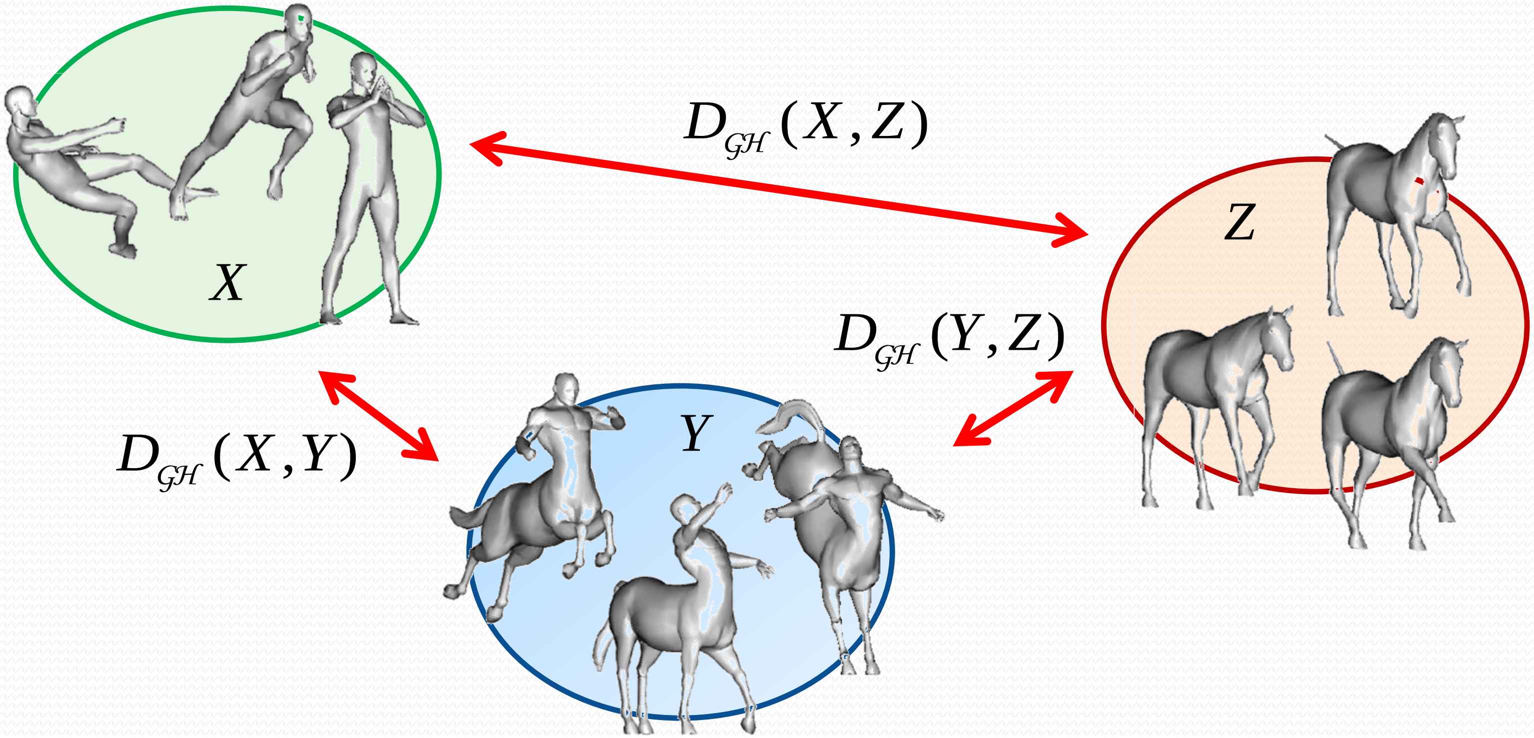
Wednesday, April 23
14:00 Room 02.09.023

Space of shapes



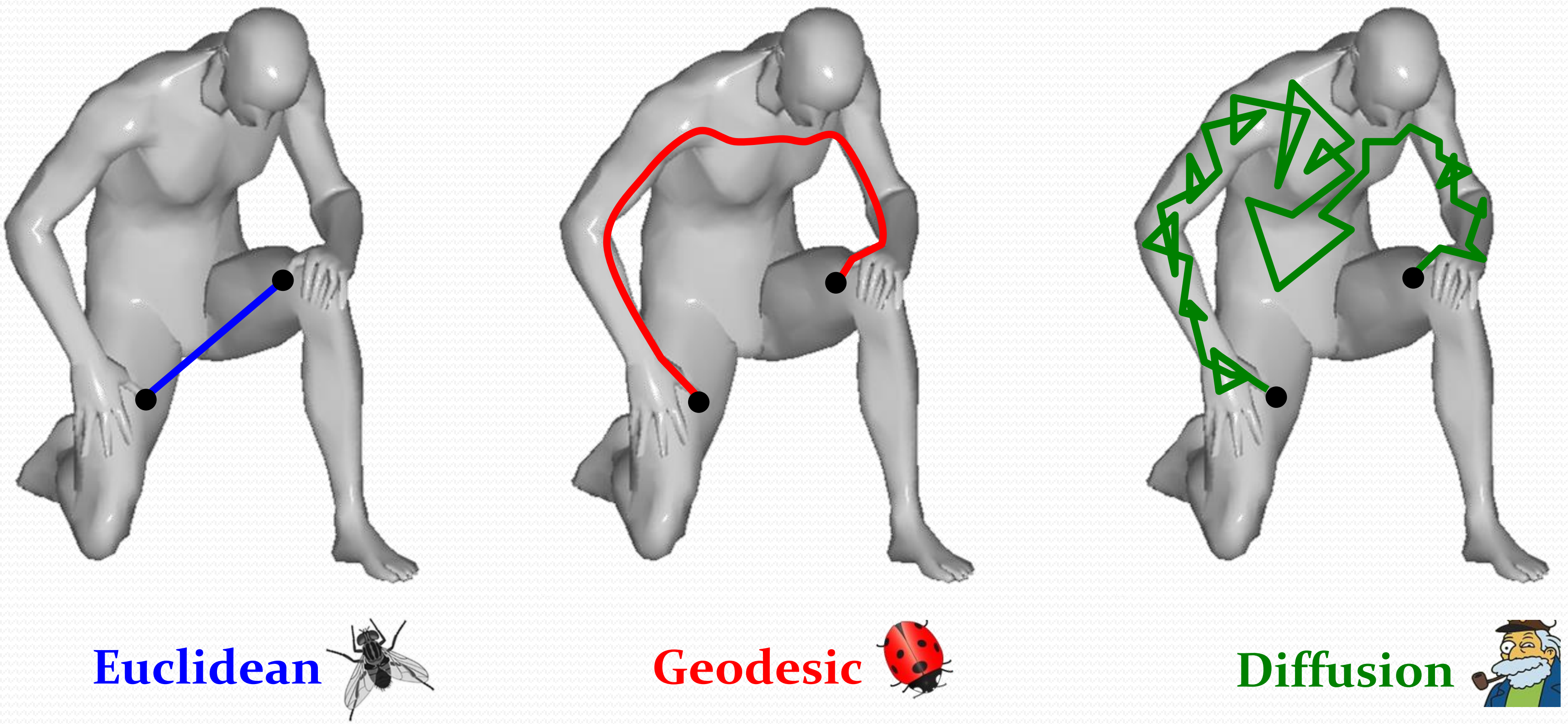
Is there something like a “space of shapes”?

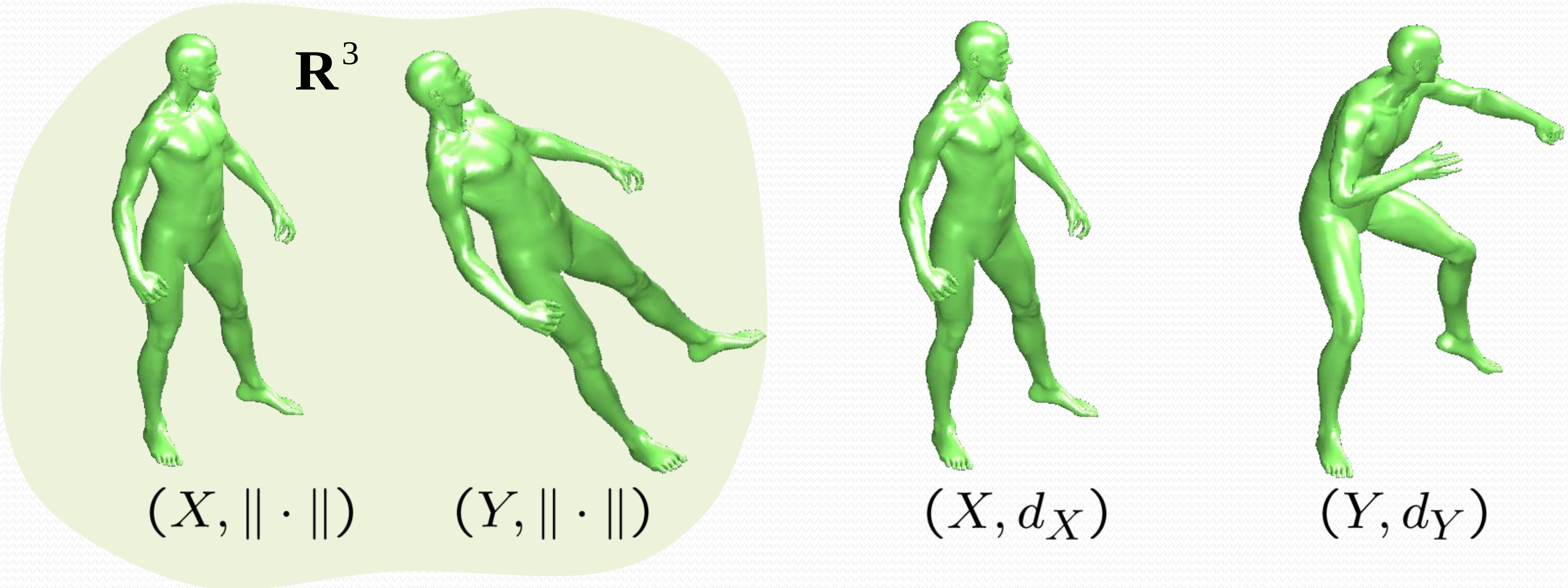
Space of shapes



Is there something like a “space of shapes”? Yes!

Choosing the metric





Rigid similarity

Part of the same metric space

Non-rigid similarity

Two different metric spaces

Main idea: Find a representation of the two shapes in a common metric space

Metric spaces

A set M is a metric space if for every pair of points $x, y \in M$ there is a metric (or distance) function $d_M : M \times M \rightarrow \mathbf{R}_+ \cup \{0, \infty\}$ such that

identity of indiscernibles $d_M(x, y) = 0 \Leftrightarrow x = y$

symmetry $d_M(x, y) = d_M(y, x)$

triangle inequality $d_M(x, y) \leq d_M(y, z) + d_M(z, x)$ for any $x, y, z \in M$

We will specify a metric space as the pair (M, d_M)

Satisfying a subset of these properties leads to the definition of "semi"-metric spaces, "pseudo"-metric spaces, etc.

Examples of metric spaces

$X = A \subset \mathbf{R}^k$ $\quad d_X(x,y) = \|x - y\|_2$

$X = $ any set $\quad d_X(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

$X = \mathbf{R}$ $\quad d_X(x,y) = |x - y|$

$d_X(x,y) = \log|x - y|$

$X = \mathbf{R}^2$ $\quad d_X((x_1,x_2),(y_1,y_2)) = \max(|x_1 - x_2|,|y_1 - y_2|)$

$X = A \times B$ $\quad d_X((a_1,b_1),(a_2,b_2)) = \sqrt{d_A(a_1,a_2)^2 + d_B(b_1,b_2)^2}$

Compactness

For the rest of this class we will assume our metric spaces to be (sequentially) compact.

A metric space (X, d_X) is *compact* if and only if every sequence in X has a Cauchy subsequence (*totally boundedness*) that converges to a point in X (*completeness*).

A sequence $\{x_n\}$ in a metric space (X, d_X) is called a *Cauchy sequence* if $d_X(x_n, x_m) \to 0$ as $n, m \to \infty$

More formally: for any $\varepsilon > 0$ there exists an n_0 such that $d_X(x_n, x_m) < \varepsilon$ whenever $n, m \geq n_0$

Compactness allows to apply many techniques of calculus on metric spaces, and has some important consequences.

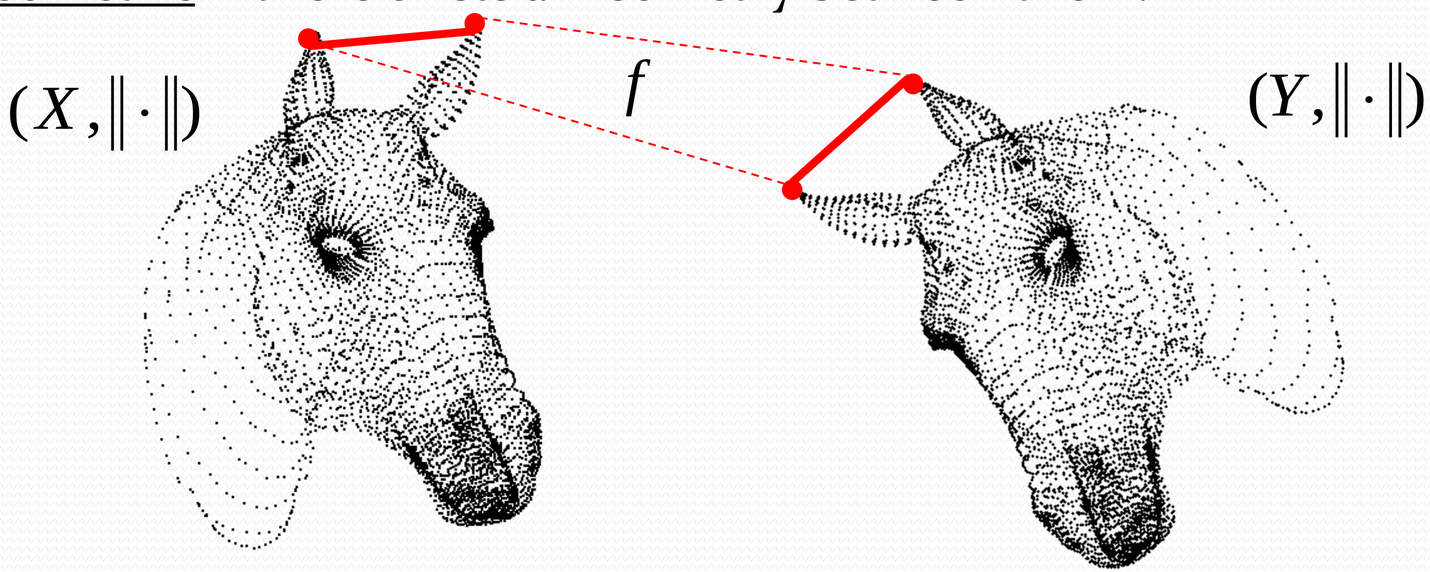
Isometries

Let (X, d_X) and (Y, d_Y) be two metric spaces.

A map $f : X \to Y$ is called distance-preserving if

$$d_X(x, y) = d_Y(f(x), f(y)) \quad \text{for any } x, y \in X$$

A bijective, distance-preserving map is called an isometry. Two spaces are isometric if there exists an isometry between them.



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Exercise: Show that any isometry is a homeomorphism.

Exercise: Isn't bijectivity redundant?

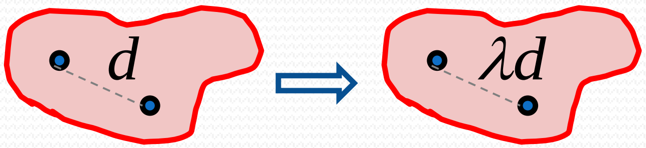
Answer: A surjective, distance-preserving map is called an isometry.

Exercise: Show that "being isometric" is an equivalence relation.

Metrics

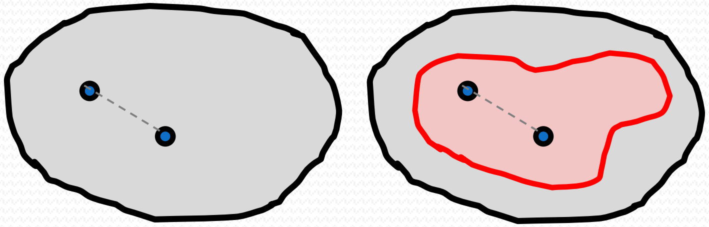
Let X be a metric space and $\lambda > 0$. The metric space λX , which we call <u>X dilated</u>, is the same set X equipped with another distance function $d_{\lambda X}$ defined by

$$d_{\lambda X}(x, y) = \lambda d_X(x, y) \quad \text{for all} \quad x, y \in X$$



If X is a metric space and $Y \subset X$, then a metric on Y can be obtained by the <u>restriction</u> $d_Y = d_X|_Y$, such that

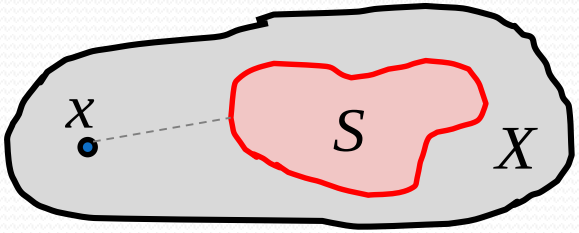
$$d_Y(x, y) = d_X(x, y) \quad \text{for all} \quad x, y \in Y$$



Metrics

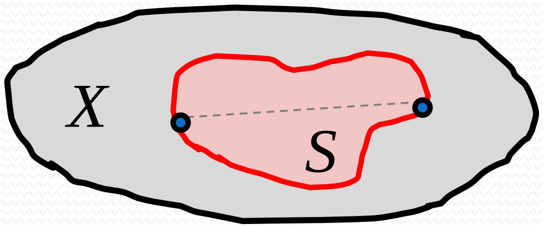
The distance from a point x to a set S in a metric space X is defined by

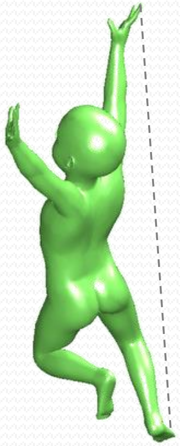
$$\mathrm{dist}_X(x,S) = \inf_{y\in S} d_X(x,y)$$



The <u>diameter</u> of a set S in a metric space X is defined by

$$\mathrm{diam}(S) = \sup_{x,y\in S} d_X(x,y)$$





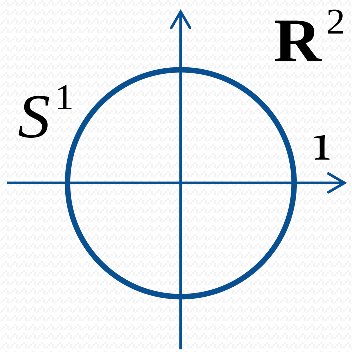
The compactness of X ensures that $\mathrm{diam}(X) < \infty$ and that there exist two points $x, y \in X$ such that $\mathrm{diam}(X) = d_X(x,y)$

Ambient space

If X is a metric space and $Y \subset X$, then X is called ambient space for Y.

Restricting d_X to $d_X|_Y$ is the simplest, but not the only way to define a metric on a subset. In many cases it is more natural to consider an *intrinsic metric*, which is generally not equal to the one restricted from the ambient space.

Example



S^1 carries the restricted Euclidean metric $\|\cdot\|$

An alternative is the (shortest) arc length γ

Question: is $(S^1, \|\cdot\|)$ isometric to (S^1, γ) ?

Lipschitz maps

A map $f : X \to Y$ between metric spaces is called Lipschitz if there exists a $C \geq 0$ such that

$$d_Y(f(x_1), f(x_2)) \leq C d_X(x_1, x_2) \qquad \text{for all} \quad x_1, x_2 \in X$$

All Lipschitz maps are continuous (exercise!).

Any suitable value of C is referred to as a Lipschitz constant of f.
The minimal Lipschitz constant is called the dilatation of f, denoted by $\operatorname{dil} f$

$$\operatorname{dil} f = \sup_{x_1, x_2 \in X} \frac{d_Y(f(x_1), f(x_2))}{d_X(x_1, x_2)}$$

Bi-Lipschitz maps

A map $f : X \to Y$ between metric spaces is called bi-Lipschitz if there are positive constants c and C such that

$$c d_X(x_1, x_2) \le d_Y(f(x_1), f(x_2)) \le C d_X(x_1, x_2)$$

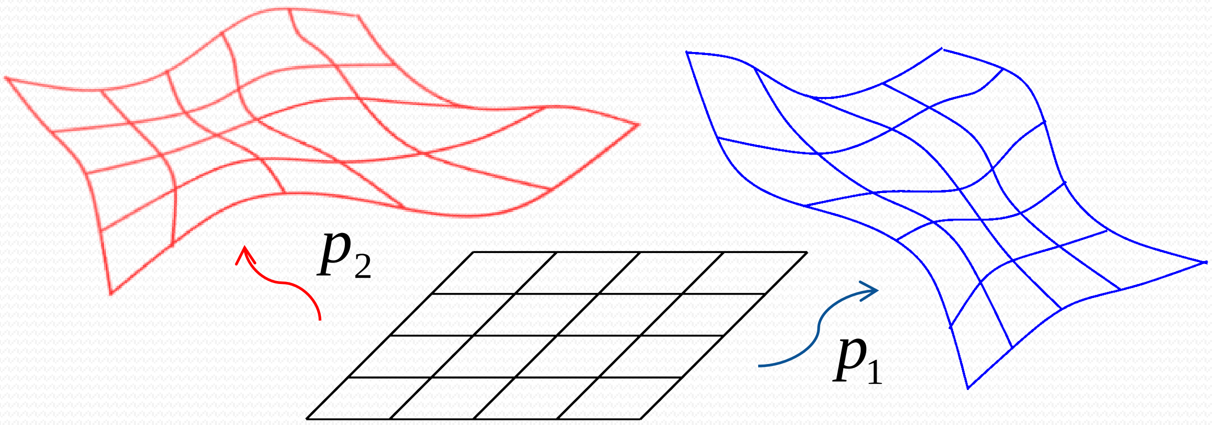
for all $x_1, x_2 \in X$

A map with Lipschitz constant $C = 1$ is called nonexpanding.

Exercise: Prove that $\mathrm{dist}_X(x, S)$ is a nonexpanding function.

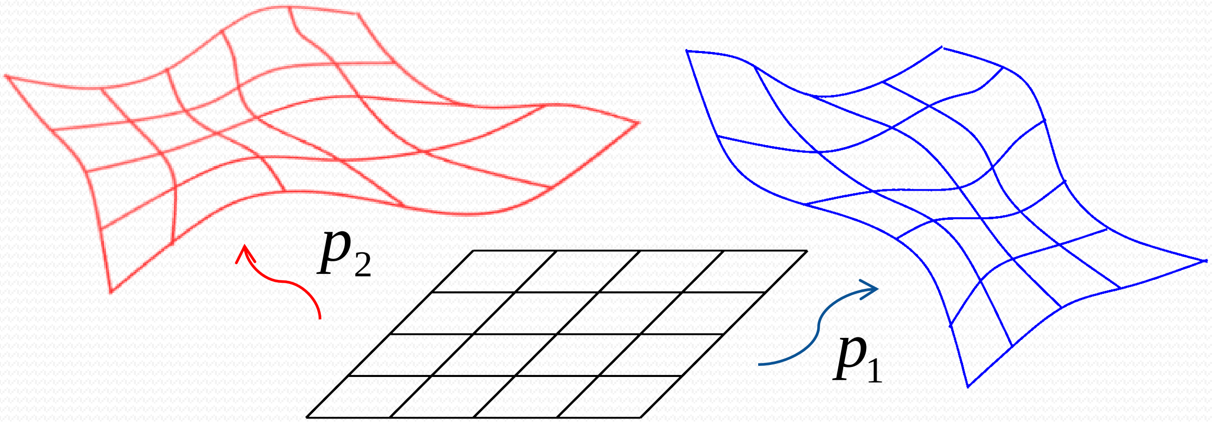
A first notion of "closeness"

Smooth surfaces in $\mathbf{R}^3$ can be (at least locally) parametrized by a domain $D \subset \mathbf{R}^2$, for example as graphs of smooth functions $h : D \to \mathbf{R}$ or as images of embeddings $p : D \to \mathbf{R}^3$



The two parametrizations determine a homeomorphism from one to the other

A first notion of “closeness”



If the surfaces are “close enough” to each other, the homeomorphism should only slightly change certain quantities such as distances, metric tensors, or their derivatives.

We can say that two spaces have a small distance between them if there is a homeomorphism which “almost preserves” certain geometric characteristic, e.g. the distance.

Lipschitz distance

The idea is to consider two metric spaces *X* and *Y* close to each other if there is a *homeomorphism* $f : X \to Y$ such that

$$\frac{d_Y(f(x_1), f(x_2))}{d_X(x_1, x_2)} \approx 1 \qquad \text{for all} \quad x_1, x_2 \in X$$

This definition gives a way to measure *relative* change between metrics.

Lipschitz distance

The idea is to consider two metric spaces X and Y close to each other if there is a *homeomorphism* $f : X \to Y$ such that

$$\frac{d_Y(f(x_1), f(x_2))}{d_X(x_1, x_2)} \approx 1 \qquad \text{for all} \quad x_1, x_2 \in X$$

Equivalently, we may require

$$\frac{d_X(x_1, x_2)}{d_Y(f(x_1), f(x_2))} \approx 1 \qquad \text{for all} \quad x_1, x_2 \in X$$

Lipschitz distance

Recall that the dilatation of a Lipschitz map f is defined by

$$\operatorname{dil} f = \sup_{x_1, x_2 \in X} \frac{d_Y(f(x_1), f(x_2))}{d_X(x_1, x_2)}$$

Since we are dealing with homeomorphisms, we get for the inverse map f^{-1}

$$\operatorname{dil} f^{-1} = \sup_{y_1, y_2 \in Y} \frac{d_X(f^{-1}(y_1), f^{-1}(y_2))}{d_Y(y_1, y_2)}$$

$$= \sup_{x_1, x_2 \in X} \frac{d_X(x_1, x_2)}{d_Y(f(x_1), f(x_2))}$$

Lipschitz distance

Let us consider the maximum relative error that we get when mapping via f

$$\varepsilon(X,Y,f) = \max\left\{ \sup_{x_1,x_2\in X} \frac{d_Y(f(x_1),f(x_2))}{d_X(x_1,x_2)}, \sup_{x_1,x_2\in X} \frac{d_X(x_1,x_2)}{d_Y(f(x_1),f(x_2))} \right\}$$
$$= \max\left\{\mathrm{dil}(f), \mathrm{dil}(f^{-1})\right\}$$

We are interested in maps yielding a relative error $\varepsilon(X,Y,f) \approx 1$.
This corresponds to requiring

$$\inf_{f:X\to Y} \log \varepsilon(X,Y,f) \to 0$$

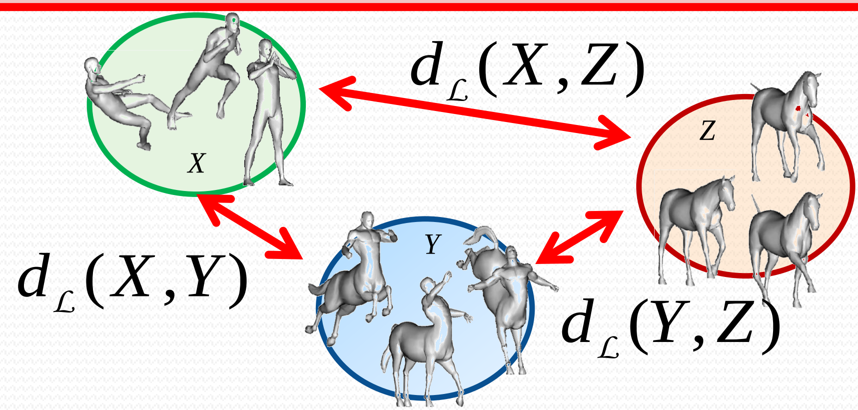
Lipschitz distance

The **Lipschitz distance** between two metric spaces X and Y is defined by

$$d_{\mathcal{L}}(X,Y) = \inf_{f:X\to Y} \log\left(\max\left\{\mathrm{dil}(f), \mathrm{dil}(f^{-1})\right\}\right)$$

The infimum is taken over all homeomorphisms such that f and f^{-1} are Lipschitz maps (bi-Lipschitz homeomorphisms).

$d_{\mathcal{L}}$ is a metric on the space of isometry classes of compact metric spaces.



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$d_{\mathcal{L}}$ is a metric on the space of isometry classes of compact metric spaces.

We set $d_{\mathcal{L}}(X,Y) = \infty$ if there are no bi-Lipschitz homeomorphisms from X to Y.

Thus, the Lipschitz distance is not suitable for comparing metric spaces that are not bi-Lipschitz homeomorphic.

Lipschitz distance

$$d_{\mathcal{L}}(X,Y) = \inf_{f:X\to Y} \log\left(\max\left\{\mathrm{dil}(f), \mathrm{dil}(f^{-1})\right\}\right)$$

<u>Non-negativity</u> $\left(d_{\mathcal{L}}(X,Y) \geq 0\right)$

$f : X \to Y$ is a homeomorphism, therefore either $\mathrm{dil}(f) \geq 1$ or $\mathrm{dil}(f^{-1}) \geq 1$ (f and f^{-1} cannot both decrease the distances).

<u>Symmetry</u> $\left(d_{\mathcal{L}}(X,Y) = d_{\mathcal{L}}(Y,X)\right)$

Trivial

Lipschitz distance

$$d_{\mathcal{L}}(X,Y) = \inf_{f:X\to Y} \log\left(\max\left\{\mathrm{dil}(f), \mathrm{dil}(f^{-1})\right\}\right)$$

<u>Triangle inequality</u> $\left(d_{\mathcal{L}}(X,Z) \le d_{\mathcal{L}}(X,Y) + d_{\mathcal{L}}(Y,Z)\right)$

$f: X \to Y,\ g: Y \to Z \quad \Longrightarrow \quad g \circ f: X \to Z$

bi-Lipschitz homeomorphisms bi-Lipschitz homeomorphism

$$\mathrm{dil}(g \circ f) \le \mathrm{dil}(f) \cdot \mathrm{dil}(g)$$

<u>Exercise</u>: Prove the above facts.

Hence $\log(\mathrm{dil}(g \circ f)) \le \log(\mathrm{dil}(f)) + \log(\mathrm{dil}(g))$, and similarly for $f^{-1} \circ g^{-1}$

This implies $d_{\mathcal{L}}(X,Z) \le d_{\mathcal{L}}(X,Y) + d_{\mathcal{L}}(Y,Z)$

Lipschitz distance

$$d_{\mathcal{L}}(X,Y) = \inf_{f:X\to Y} \log\left(\max\left\{\text{dil}(f), \text{dil}(f^{-1})\right\}\right)$$

<u>Identity of indiscernibles</u> $\left(d_{\mathcal{L}}(X,Y) = 0 \Leftrightarrow X = Y\right)$

$X = Y \Rightarrow d_{\mathcal{L}}(X,Y) = 0$ — *X* and *Y* are isometric by assumption, thus substituting an isometry $f : X \to Y$ in the definition yields $d_{\mathcal{L}}(X,Y) = 0$

$d_{\mathcal{L}}(X,Y) = 0 \Rightarrow X = Y$ — *Sketch of proof* (next page)

Lipschitz distance

$$d_{\mathcal{L}}(X,Y) = \inf_{f:X\to Y} \log(\max\{\mathrm{dil}(f), \mathrm{dil}(f^{-1})\})$$

$$d_{\mathcal{L}}(X,Y) = 0 \Rightarrow X = Y$$

$d_{\mathcal{L}}(X,Y) = 0$ implies that there exists a sequence of maps $f_n : X \to Y$ such that $\mathrm{dil}(f_n) \to 1$ and $\mathrm{dil}(f_n^{-1}) \to 1$ as $n \to \infty$ (this comes from compactness)

The sequence f_n converges to f (this comes from compactness).

Then we have for all $x, x' \in X$, $d_Y(f_n(x), f_n(x'))/d_X(x,x') \to 1$ and hence:

$$d_Y(f(x), f(x')) = d_X(x,x')$$

This means that f is distance-preserving, and similarly for $g : Y \to X$.

The composition $f \circ g$ is distance-preserving, and bijective by compactness of Y.

Hence, f is surjective and thus an isometry.

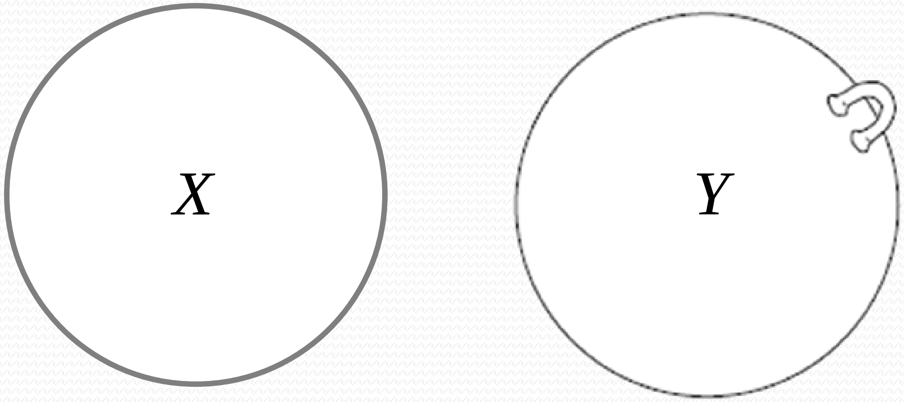
Lipschitz distance

Note that we needed compactness of X and Y in order to prove

$$d_{\mathcal{L}}(X,Y)=0 \Leftrightarrow X=Y$$

Lipschitz distance

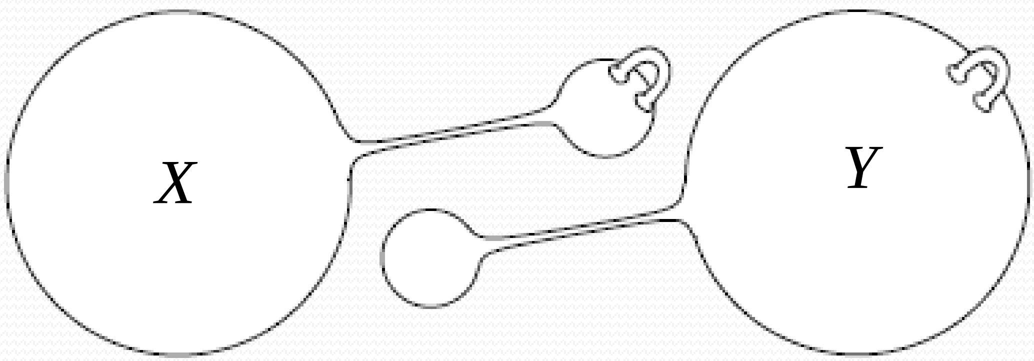
Disadvantage: It requires spaces to be homeomorphic.



$$d_{\mathcal{L}}(X, Y) = \infty$$

Lipschitz distance

Disadvantage: Even for two homeomorphic spaces, it may happen that the "similarity" is not realized by a homeomorphism.



Any homeomorphism (in fact, any continuous map) from X to Y essentially distorts distances between some points.

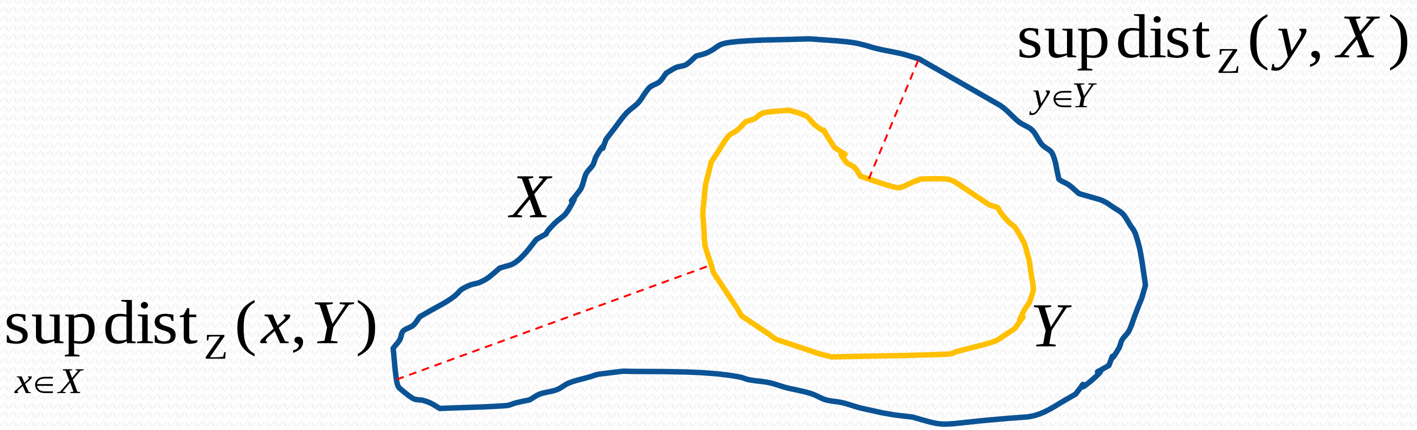
Intuitively, we see that the distance between X and Y should be small because each of them is contained in a small neighborhood of the other in $\mathbf{R}^3$

Hausdorff distance

The **Hausdorff distance** between two compact subsets $X, Y \subset (Z, d_Z)$ is defined by

$$d_{\mathcal{H}}^{Z}(X,Y) = \max\left\{\sup_{x\in X} \mathrm{dist}_Z(x,Y), \sup_{y\in Y} \mathrm{dist}_Z(y,X)\right\}$$

$d_{\mathcal{H}}^{Z}$ is a semi-metric on the space of compact subsets of a metric space.



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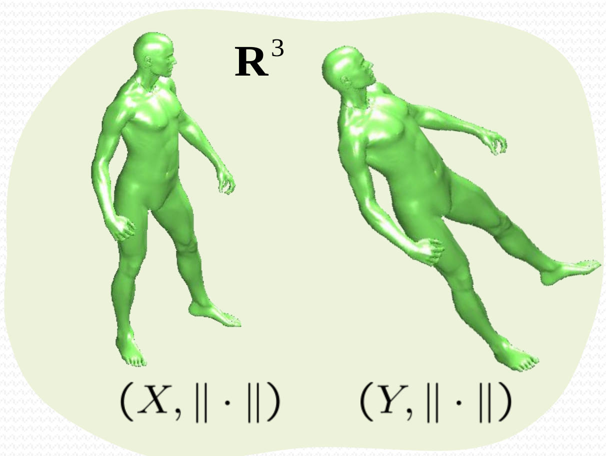

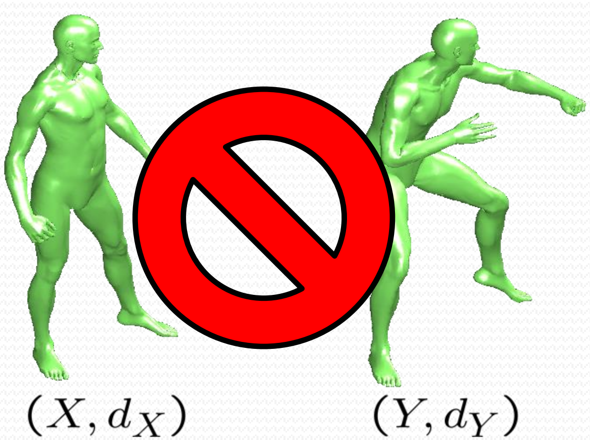

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$d_{\mathcal{H}}^{Z}$ is a semi-metric on the space of compact subsets of a metric space.

Being a "semi"-metric means that $d_{\mathcal{H}}^{Z}(X,Y) = 0 \Leftrightarrow X = Y$ does not hold.

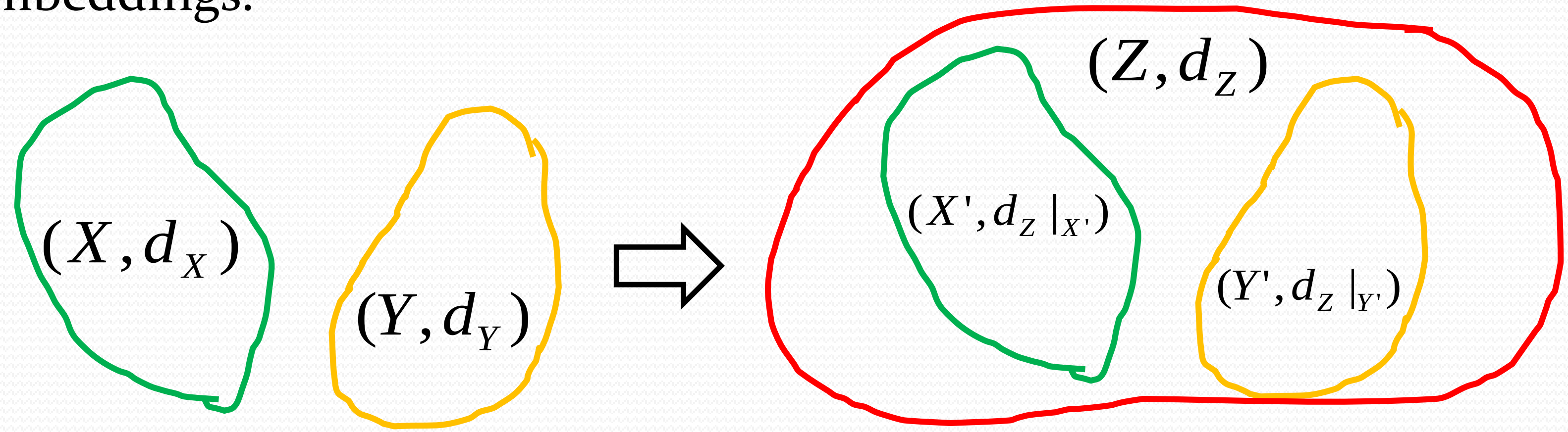
Exercise: Show that $\mathrm{dist}_Z(x, \overline{A}) = 0$ for all $x \in A \subset X$, and $\mathrm{dist}_Z(x, A) = 0$ for all $x \in \overline{A}$, where $\overline{A}$ denotes the closure of A.

Note that a difference in a single point can make $d_{\mathcal{H}}^{Z}$ arbitrarily large!

Gromov-Hausdorff distance

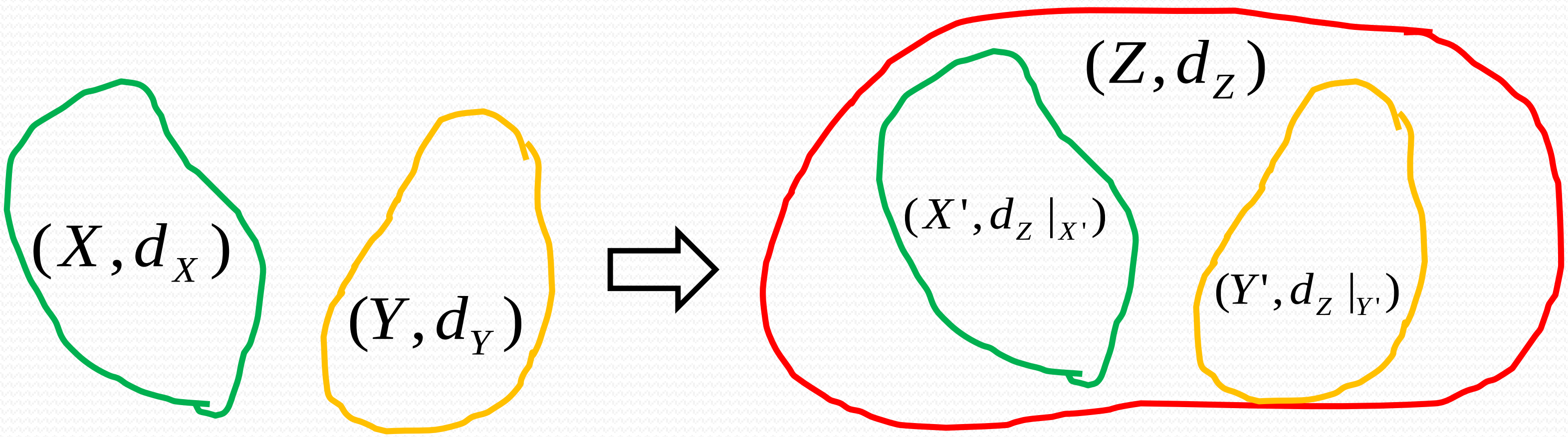
Can we define a Hausdorff distance *between metric spaces*?

The general idea is to *embed* the two metric spaces *X* and *Y* into a new metric space *Z*, and compute the Hausdorff distance on the resulting embeddings.



We proceed by requiring $d_{\mathcal{GH}}(X,Y) < r$ for $r > 0$ if and only if there exists a metric space Z and subspaces $X', Y' \subset Z$ which are isometric to *X* and *Y*, and such that $d_{\mathcal{H}}(X', Y') < r$.

Gromov-Hausdorff distance



We will indeed <u>define</u> $d_{\mathcal{GH}}(X,Y)$ as the minimum *r* for which such *Z*, *X'* and *Y'* exist.

$$\begin{array}{ccc} (X, d_X) & \overset{d_{\mathcal{GH}}}{\Longleftrightarrow} & (Y, d_Y) \\ f \downarrow & & \downarrow g \\ (f(X), d_Z|_{X'}) & \overset{d_{\mathcal{H}}^{Z}}{\Longleftrightarrow} & (g(Y), d_Z|_{Y'}) \end{array}$$

Gromov-Hausdorff distance

The **Gromov-Hausdorff distance** between two metric spaces X and Y is defined by

$$d_{\mathcal{GH}}(X,Y) = \inf_{Z,f,g} d_{\mathcal{H}}^{Z}(f(X), g(Y))$$

The infimum is taken over all ambient spaces Z and isometric embeddings (distance preserving) $f : X \to Z,\ \ g : Y \to Z$

$d_{\mathcal{GH}}$ is a metric on the space of isometry classes of compact metric spaces.

All ambient metric spaces Z is indeed a huge class of metric spaces!

Example: rigid isometries

Let us consider the case in which (X, d_X) and (Y, d_Y) are subsets of a larger metric space (this brings us back to the Hausdorff case). For example, take $X, Y \subset (\mathbf{R}^3, \|\cdot\|)$.

$$d_{\mathcal{GH}}(X,Y) = \inf_{Z,f,g} d_{\mathcal{H}}^{Z}(f(X), g(Y))$$

$$\Downarrow$$

$$d_{\mathcal{GH}}^{\text{rigid}}(X,Y) = \inf_{\phi: Z \to Z} d_{\mathcal{H}}^{Z}(X, \phi(X))$$

Where ϕ sweeps all rigid isometries of the form $\phi(\cdot) = R(\cdot) + T$ with $\det R = \pm 1$

Example: rigid isometries

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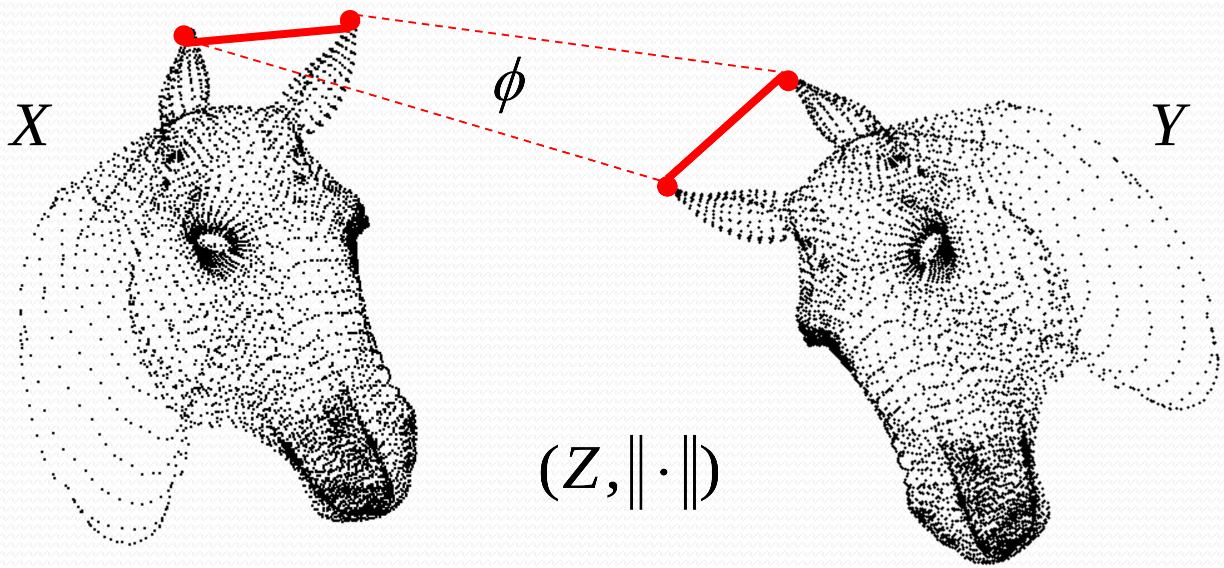


Correspondence

We will not prove the metric axioms on $d_{\mathcal{GH}}$ (yay!), but let us try to give a more "computational friendly" formulation.

A **correspondence** between two sets X and Y is a set $R \subset X \times Y$ satisfying:

- for every $x \in X$ there exists at least one $y \in Y$ such that $(x, y) \in R$
- for every $y \in Y$ there exists at least one $x \in X$ such that $(x, y) \in R$

What we are going to prove is:

$d_{\mathcal{GH}}(X, Y) < r$ if and only if there is a correspondence between X and Y such that **if** $x, x' \in X$ and $y, y' \in Y$ are corresponding pairs of points, **then** $|d_X(x, x') - d_Y(y, y')| < 2r$

Correspondence

Any surjective map $f: X \to Y$ defines a correspondence

$$R = \{(x, f(x)) : x \in X\}$$

Note, however, that not every correspondence is associated with a map! We can regard a correspondence as a “multi-valued” map, in which a single point is allowed to have more than one image.

One way to sidestep this issue is by using an auxiliary set. Let $f: Z \to X$ and $g: Z \to Y$ be two surjective maps from some “reference” set Z. Then we can define a correspondence as

$$R = \{(f(z), g(z)) : z \in Z\}$$

Metric distortion

Let (X, d_X) and (Y, d_Y) be (compact) metric spaces and $f : X \to Y$ an arbitrary (even noncontinuous) map. The distortion of f is defined by

$$\operatorname{dis} f = \sup_{x_1, x_2 \in X} \left| d_Y(f(x_1), f(x_2)) - d_X(x_1, x_2) \right|$$

Distortion measures the *absolute* change of distances.

Compare with the requirement we gave in the Lipschitz case:

$$\frac{d_Y(f(x_1), f(x_2))}{d_X(x_1, x_2)} \approx 1 \qquad \text{for all} \quad x_1, x_2 \in X$$

Metric distortion

The distortion of a correspondence $R \subset (X, d_X) \times (Y, d_Y)$ is defined by

$$\operatorname{dis} R = \sup\{|d_X(x, x') - d_Y(y, y')| : (x, y), (x', y') \in R\}$$

Note that $\operatorname{dis} f = \operatorname{dis} R$ for any surjective map $f : X \to Y$, where R is the associated correspondence $R = \{(x, f(x)) : x \in X\}$

The key result is that $\operatorname{dis} R = 0$ if and only if R is associated with an isometry.

We say that f is an ε-nearisometry if $\operatorname{dis} f \le \varepsilon$

Gromov-Hausdorff distance

The **Gromov-Hausdorff distance** between two metric spaces *X* and *Y* is defined by

$$d_{\mathcal{GH}}(X,Y) = \frac{1}{2} \inf_R \operatorname{dis} R$$

The infimum is taken over all correspondences *R* between *X* and *Y*.

$d_{\mathcal{GH}}$ is a metric on the space of isometry classes of compact metric spaces.

Note that $d_{\mathcal{GH}}(X,Y) = 0$ if and only if *X* and *Y* are isometric.
In addition, it is a *finite* quantity (differently from the Lipschitz distance).

Gromov-Hausdorff distance

What we are going to prove is:

$d_{GH}(X,Y) < r$ if and only if there is a correspondence between X and Y such that **if** $x, x' \in X$ and $y, y' \in Y$ are corresponding pairs of points, **then** $|d_X(x,x') - d_Y(y,y')| < 2r$

The equivalence between the two formulations must be proven!

We will indeed <u>define</u> $d_{GH}(X,Y)$ as the minimum r for which such Z, X' and Y' exist.

With the new formulation, the GH distance is equal to the infimum of $r > 0$ for which there exists a correspondence with $\operatorname{dis} R < 2r$

Gromov-Hausdorff distance

$$d_{\mathcal{GH}}(X,Y) = \frac{1}{2} \inf_{R} \operatorname{dis} R$$

This notion of distance encodes the metric disparity between the metric spaces in a computationally impractical way.

Coverings

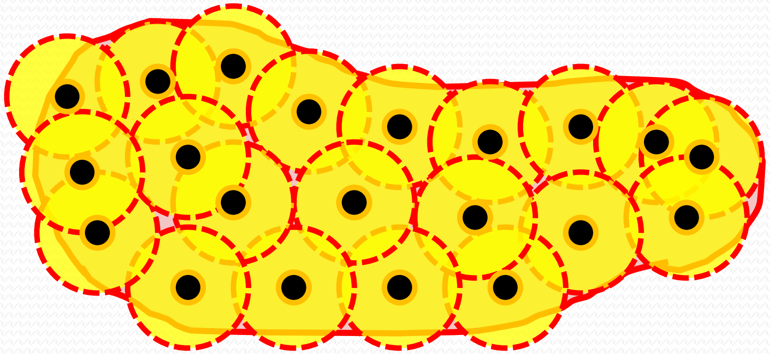
Let $x \in (X, d_X)$. An open ball of radius $r > 0$ centered at x is defined by

$$B_X(x,r) = \{z \in X : d_X(x,z) < r\}$$

For a subset A of X, we define

$$B_X(A,r) = \bigcup_{a \in A} B_X(a,r)$$

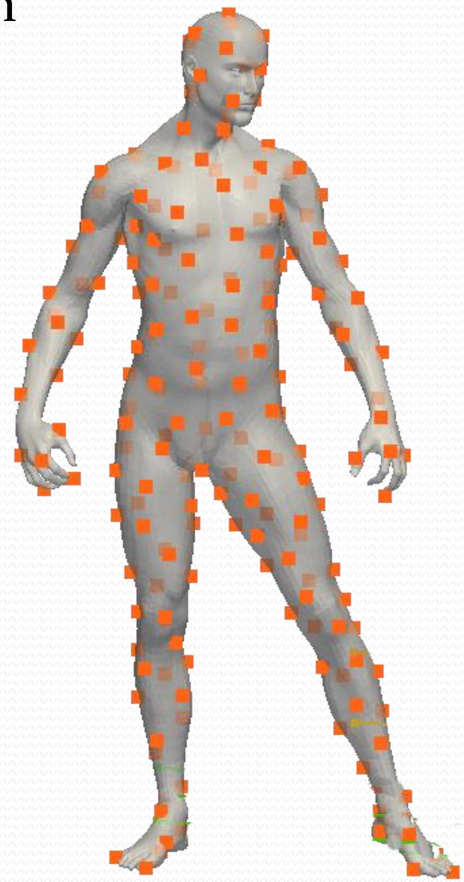
A set $C \subset X$ is an r-covering of X if $B_X(C,r) = X$



Coverings

Let $\{x_1,\ldots,x_n\} \subset X$ be a r-covering of the compact metric space (X, d_X)
Then

$$d_{\mathcal{GH}}(X, \{x_1,\ldots,x_n\}) \le r$$



This tells us that "shape samplings" are close to the underlying shapes in the Gromov-Hausdorff sense

Coverings

Let $\{x_i\}_{i=1}^{m}$ be a *r*-covering of X, and $\{y_j\}_{j=1}^{m'}$ be a *r'*-covering of Y. Then

$$\left| d_{\mathcal{GH}}(X,Y) - d_{\mathcal{GH}}(\{x_i\}_{i=1}^{m}, \{y_j\}_{j=1}^{m'}) \right| \le r + r'$$

$d_{\mathcal{GH}}$ is consistent to sampling

If we have a way to compute $d_{\mathcal{GH}}$ for dense enough (small *r'*) samplings of *X* and *Y*, then it would give us a good approximation to what happens in the continuous spaces.

This gives a formal justification for the surface recognition problem from point samples, showing that it is well posed.

Coverings

Tighter bounds can be computed depending on the algorithm.
Recall for the rigid case

$$d_{\mathcal{GH}}^{\mathrm{rigid}}(X,Y) = \inf_{\phi:Z\to Z} d_{\mathcal{H}}^{Z}(X,\phi(X))$$

For example, there is an algorithm with provable bounds:

$$\underbrace{d_{\mathcal{H}}^{\mathrm{rigid}}(X,Y)}_{\text{unknown}} - (r+r') \le \underbrace{d_{\mathcal{H}}^{\mathrm{rigid}}\left(\{x_i\}_{i=1}^{m}, \phi(\{y_j\}_{j=1}^{m'})\right)}_{\text{observed (computable)}} \le 10\left(\underbrace{d_{\mathcal{H}}^{\mathrm{rigid}}(X,Y)}_{\text{unknown}} + (r+r')\right)$$

A computational approach

We want to compute a correspondence $R \subset X \times Y$ minimizing

$$d_{\mathcal{GH}}(X,Y) = \frac{1}{2}\inf_R \operatorname{dis} R$$

Let us rewrite

$$\begin{aligned} d_{\mathcal{GH}}(X,Y) &= \frac{1}{2}\inf_R \operatorname{dis} R \\ &= \frac{1}{2}\inf_R \sup\left\{\left|d_X(x,x') - d_Y(y,y')\right| : (x,y),(x',y') \in R\right\} \\ &= \frac{1}{2}\inf_{f:X\to Y} \sup_{x,x'\in X}\left|d_Y(f(x),f(x')) - d_X(x,x')\right| \end{aligned}$$

where f ranges over all surjective mappings from X to Y

A computational approach

$$d_{\mathcal{GH}}(X,Y) = \frac{1}{2} \inf_{f:X\to Y} \sup_{x,x'\in X} \left| d_Y(f(x), f(x')) - d_X(x,x') \right|$$

Let $\mathbf{X} = \{x_i\}_{i=1}^{m}$ be a r-covering of X, and $\mathbf{Y} = \{y_j\}_{j=1}^{m'}$ be a r'-covering of Y. Then we can define an alternative distance

$$d_{\mathcal{P}}(\mathbf{X},\mathbf{Y}) = \frac{1}{2} \min_{\pi\in P_n} \max_{1\le i,j\le n} \left| d_{\mathbf{Y}}(y_{\pi i}, y_{\pi j}) - d_{\mathbf{X}}(x_i, x_j) \right|$$

where P_n is the set of all permutations of $\{1,\ldots,n\}$.

- A permutation π provides the correspondence between the two sets
- The error term gives the pairwise distance once this correspondence has been assumed.

A computational approach

It should be evident that

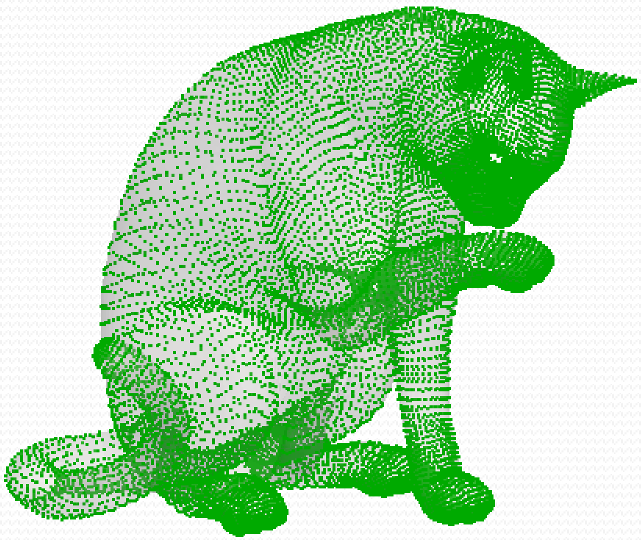
$$d_{\mathcal{GH}}(X,Y) \leq d_{\mathcal{P}}(\mathbf{X}, \mathbf{Y})$$

One can also prove

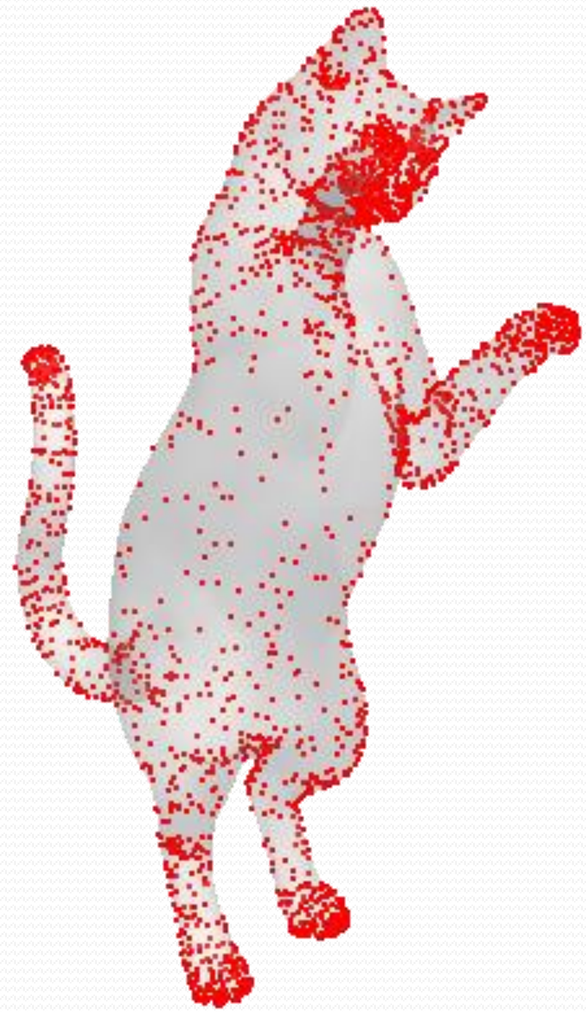
$$d_{\mathcal{GH}}(X,Y) \leq r + r' + d_{\mathcal{P}}(\mathbf{X}, \mathbf{Y})$$

The general idea now is to define coverings **X**, **Y** that provide a tighter bound on the Gromov-Hausdorff distance.

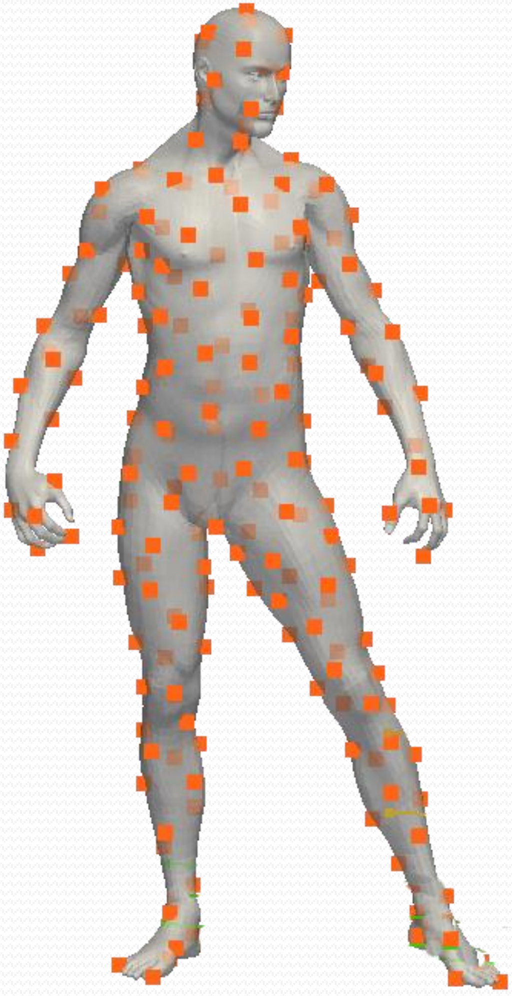
Coverings



Can we devise an optimal sampling scheme in a metric sense?



Farthest point sampling



Fix n the number of points we want to have in our final covering $\mathbf{X}_n$

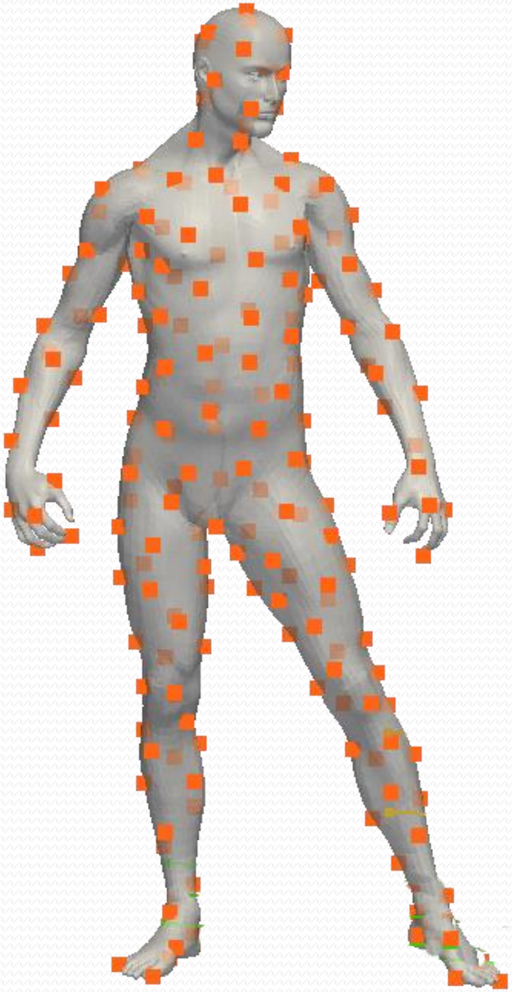
We proceed recursively.
Given $\mathbf{X}_{k-1}$, select $p \in (X, d_X)$ such that

$$p = \arg\max_{x \in X} d_X(x, \mathbf{X}_{k-1})$$

In general the maximum is not unique, one could consider all of them or randomly pick one.

Set $\mathbf{X}_k = \mathbf{X}_{k-1} \cup \{p\}$, and repeat.

Farthest point sampling



Clearly we expect different samplings depending on the starting point $\mathbf{X}_1 = \{q\}$

One way to get a more stable sampling is by setting $\mathbf{X}_2 = \{p, q\}$ such that

$$(p,q) = \arg\max_{(p,q)\in X\times X} d_X(p,q)$$

In other words, select two points attaining $\operatorname{diam}(X)$

Note however, that such a $\mathbf{X}_2$ is still in general not unique.

Voronoi sampling

The sampling $\{x_i\}_{i=1}^{n}$ represents a region $V_i \subset X$ as a single point $x_i \in X$:

$$V_i(\mathbf{X}) = \{x \in X : d_X(x, x_i) < d_X(x, x_j), x_{j \neq i} \in \mathbf{X}\}$$

This region is also known as Voronoi region.

The Voronoi decomposition replaces $x \in X$ with the closest point $\tilde{x}(x) \in \mathbf{X}$

Its *representation error* can be quantified by

$$\varepsilon(\mathbf{X}) = \operatorname{var}\{d_X(x, \tilde{x}(x))\}$$

The optimal sampling is $\arg\min_{\mathbf{X}} \varepsilon(\mathbf{X})$

Farthest point sampling

Alternatively, the optimal sampling is the one minimizing the *maximum cluster radius*

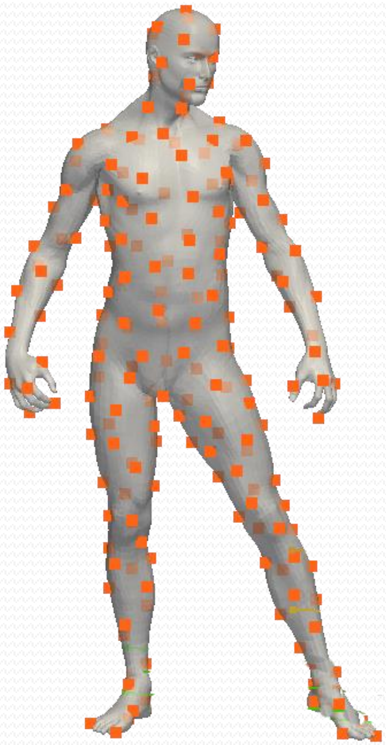
$$\varepsilon_{\infty}(\mathbf{X}) = \max_{i=1,\ldots,n} \max_{x \in V_i} d_X(x, x_i)$$

Both error criteria are **NP-hard** to compute!

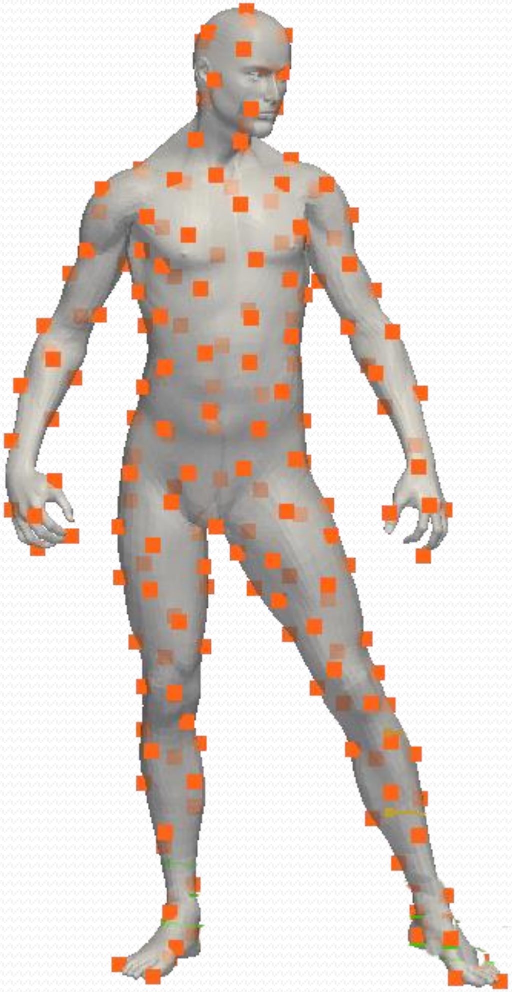
Farthest point sampling

Theorem: FPS is “almost” optimal, in the sense

$$\varepsilon_\infty(\mathbf{X}_{\text{fps}}) \leq 2 \min_{\mathbf{X}} \varepsilon_\infty(\mathbf{X})$$



Farthest point sampling



- The final sampling has *progressively increasing density.*

- *It is efficient* (provided the chosen metric is efficient to compute). Time complexity is *O(mn)*, where $m = |X|$. It can be reduced using efficient data structures.

- It is worse than optimal sampling by at most a factor of 2.

Seminar

"The metric approach to shape matching"

Alfonso Ros

Wednesday, May 28

14:00 Room 02.09.023

Suggested reading

The references below contain more than needed for the course, but cover all the key notions we have seen in this class.

- *Gromov-Hausdorff distances in Euclidean spaces*. F.Mémoli. Proc. NORDIA 2008. Sections 1 to 3.1.
- *Comparing point clouds*. F.Mémoli and G.Sapiro. Proc. SGP 2004. Sections 1, 2, 2.1, 2.2, 3.3
- *On the use of Gromov-Hausdorff distances for shape comparison*. F.Mémoli. Proc. SGP 2007. Sections 1, 2, 4
- *Numerical geometry of non-rigid shapes (Bronstein, Bronstein, Kimmel)* – Chapters 10.1, 10.2, 10.3
- *A course in metric geometry (Burago, Burago, Ivanov)* – Chapters 1.1 to 1.5, 7.1, 7.2, 7.3