Analysis of Three-Dimensional Shapes
(IN2238, TU München, Summer 2014)

Differential Geometry I
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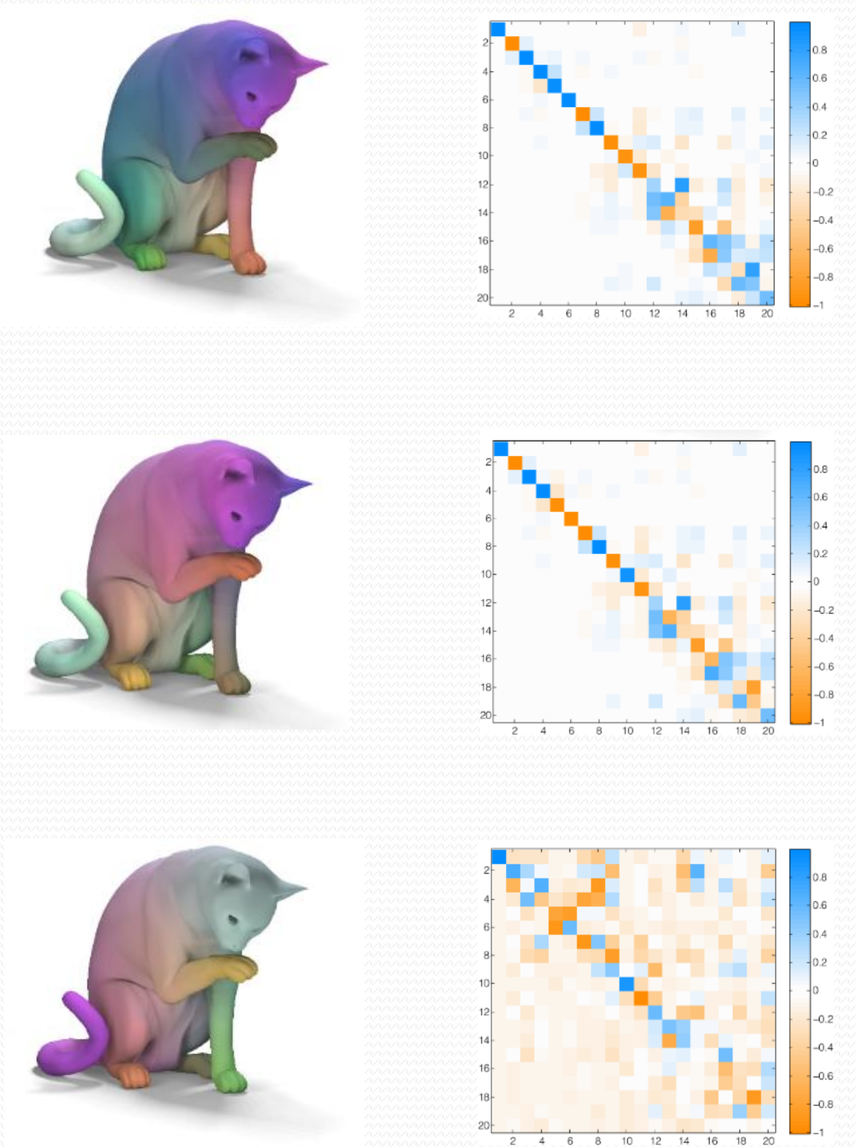
Seminar

"Functional Maps"

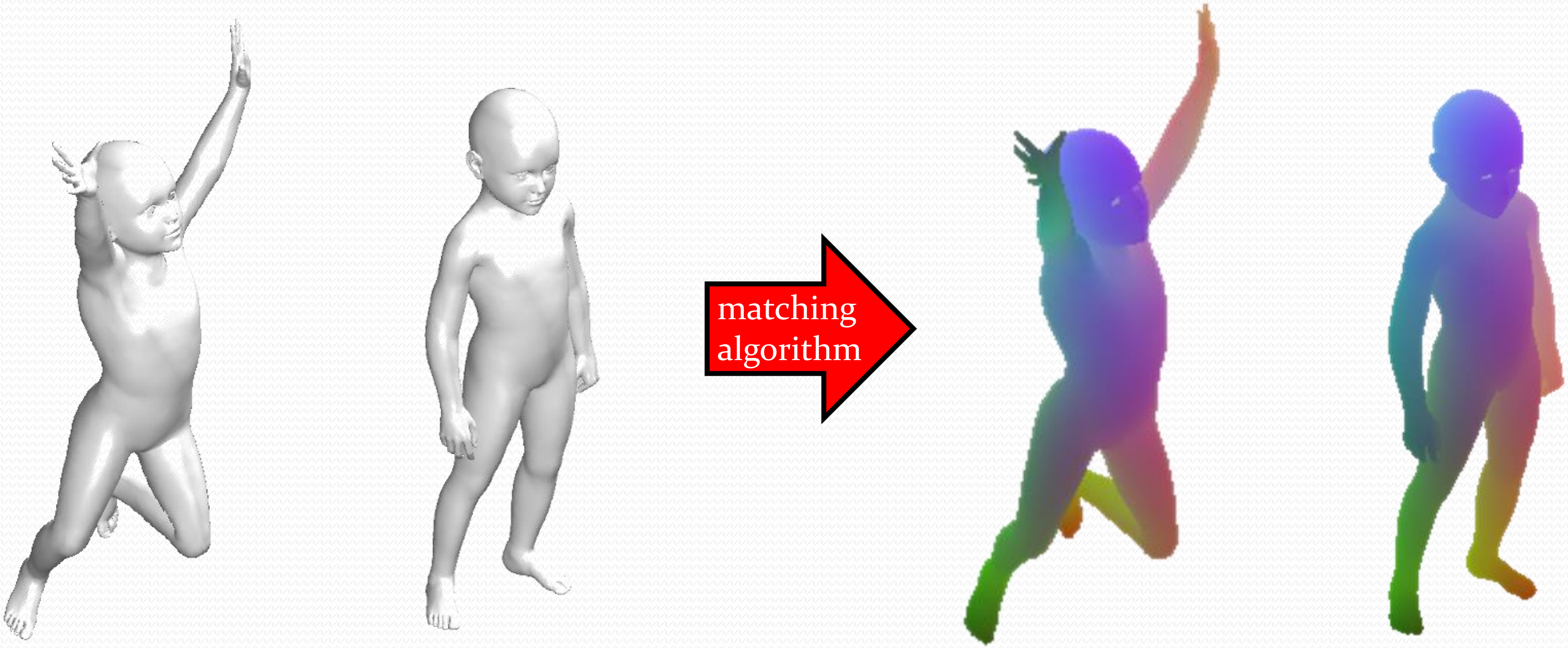
Markus Sterflinger

Wednesday, May 14th

14:00 Room 02.09.023



The matching game

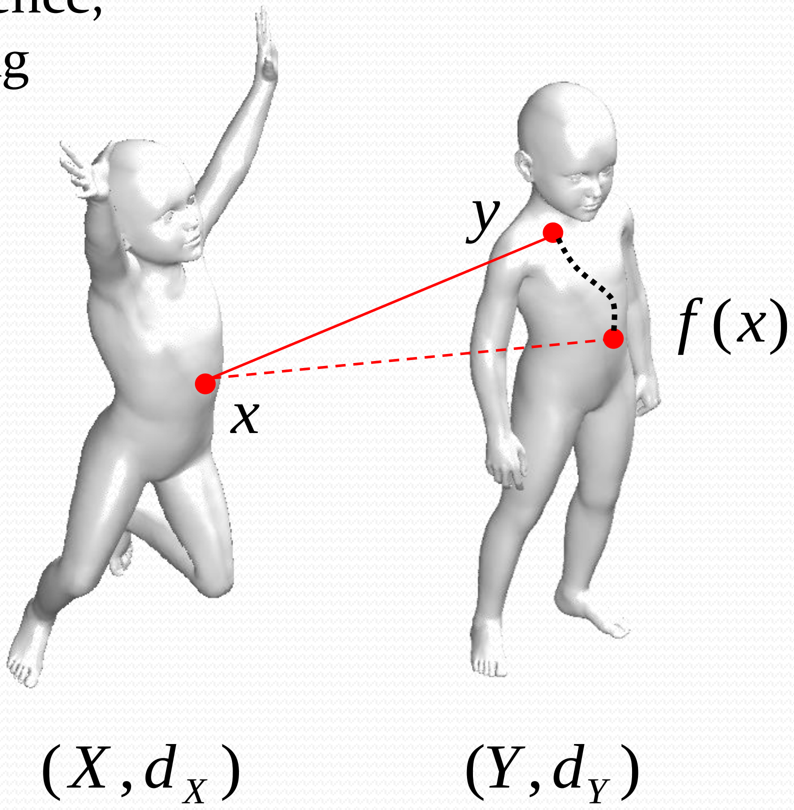


The matching game

Let $C \subset X \times Y$ be the computed correspondence, and $f : X \to Y$ be the ground-truth mapping among the two shapes (which we have).

The *average geodesic error* of C is defined by

$$\varepsilon(C) = \frac{1}{|C|} \sum_{(x,y)\in C} d_Y(y, f(x))$$



The matching game

Let A and B be the number of matched points in X and Y respectively, and let N be the total number of points.

We compute the *score* of C as:

$$\mathrm{score}(C) = \frac{A+B}{2N}\,\frac{1-\varepsilon(C)}{\mathrm{diam}\,Y}$$

$N = 59727$

$\mathrm{diam}\,Y = 119.83$

$A = 100\%$

$B = 36.64\%$

$\varepsilon = 8.81$

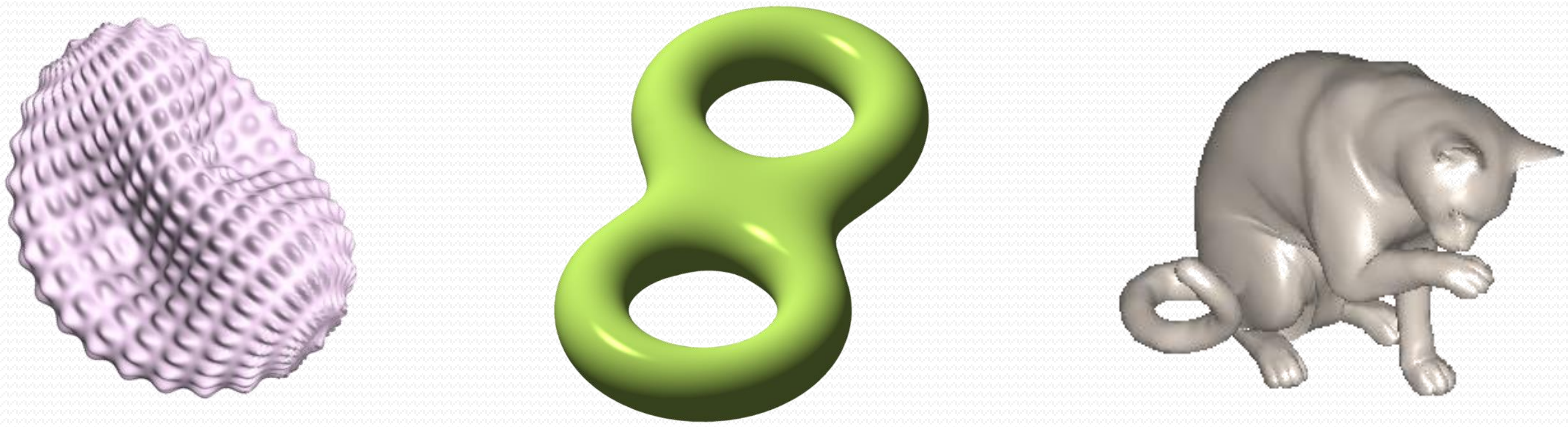
$\mathrm{score} = 0.633$

Overview

- Parametrized surfaces and first fundamental form
- Functions defined on surfaces
- Laplace-Beltrami operator
- Extension to triangulated manifolds

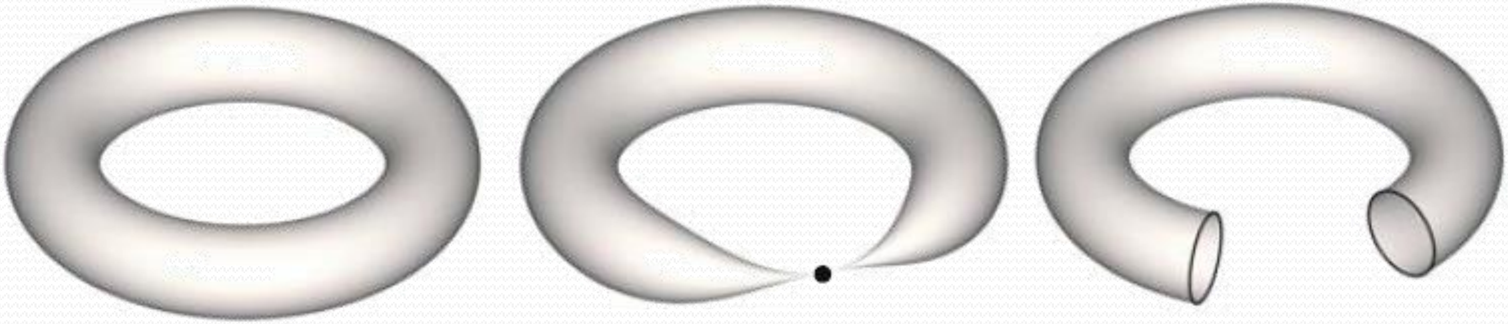
Manifolds without boundary

We will consider *2-dimensional manifolds without boundary*.



Note that this is but one particular choice. For example, we could instead model our shapes as 3-dimensional manifolds *with* boundary (interior+surface).

Some examples

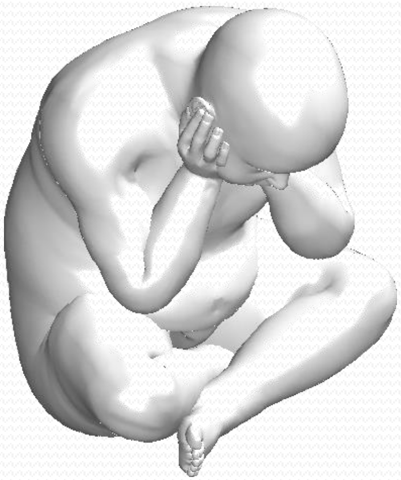


2-dimensional manifold *without* boundary

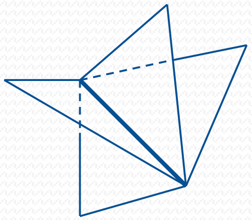
not a manifold

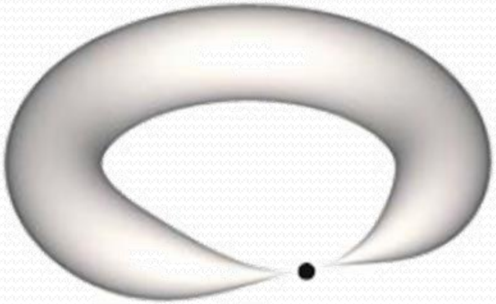
2-dimensional manifold *with* boundary (the boundary itself is a 1-dimensional manifold)

Non-manifolds

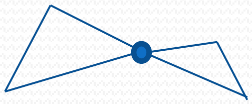


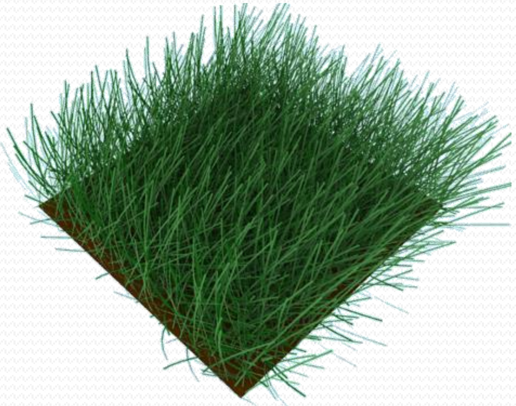
self-intersecting geometry



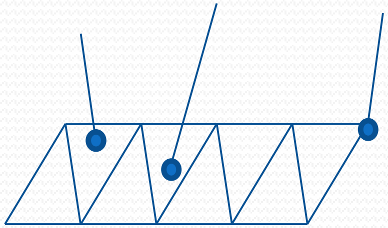


topological noise

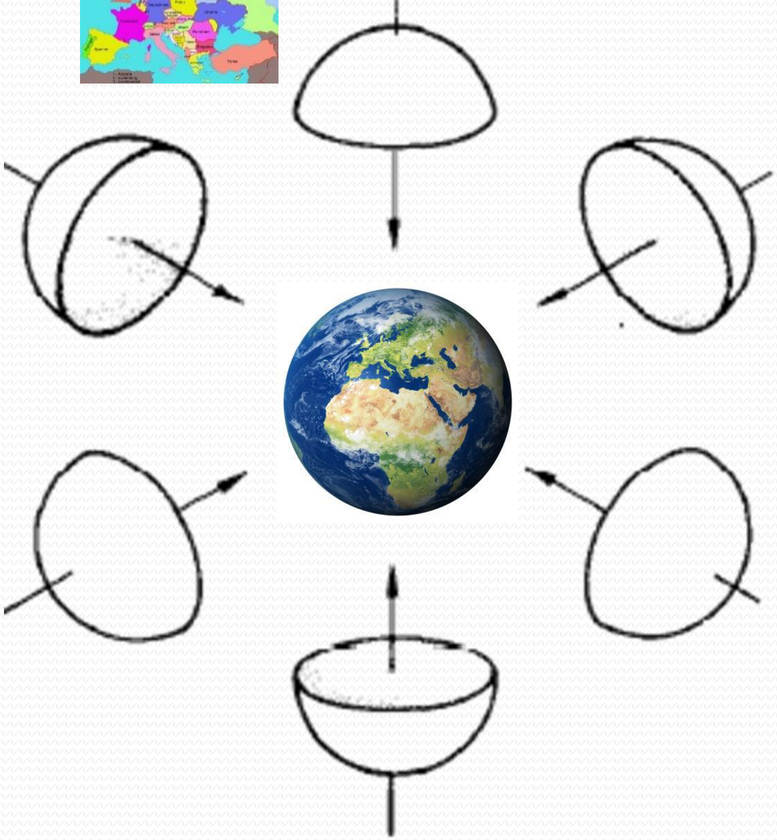




lower-dimensional structures



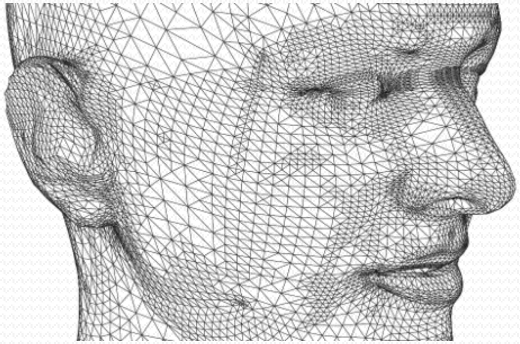
This is your world



Regular surfaces

Intuitively: A **regular surface** in $\mathbf{R}^3$ is obtained by taking pieces of a plane, deforming them, and arranging them so that the resulting shape has no sharp points, edges, or self-intersections.

This way, it makes sense to speak of *tangent planes*, and the figure is *smooth* enough so that the usual notions of calculus can be extended to it.



Parametrized curves

A **parametrized differentiable curve** is a differentiable map $\alpha : I \to \mathbf{R}^3$ of an open interval $I = (a,b)$ of the real line into $\mathbf{R}^3$.

$$\alpha(t) = (x(t), y(t), z(t))$$

- t is called *parameter*
- $x(t)$, $y(t)$, $z(t)$ are differentiable

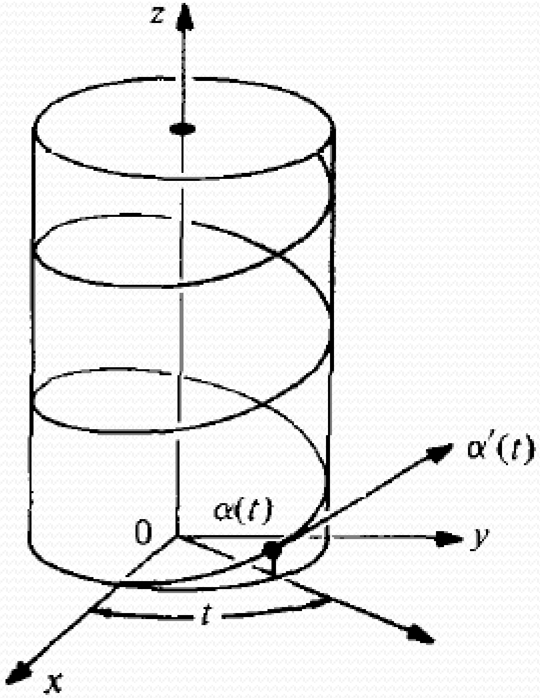
The **tangent vector** (or velocity vector) of the curve at t is defined as:

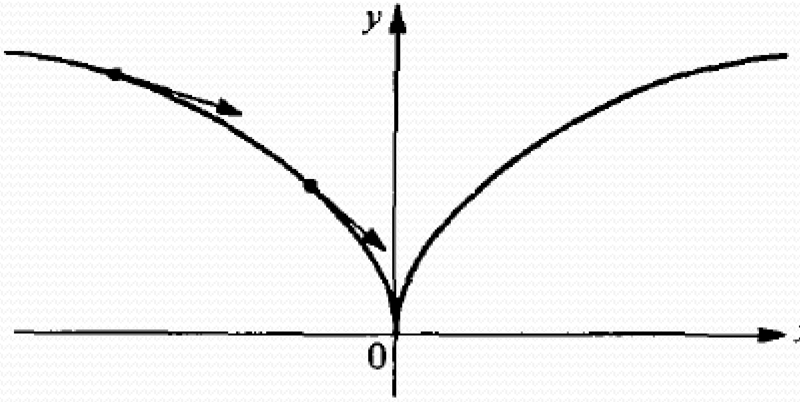
$$\alpha'(t) = (x'(t), y'(t), z'(t))$$

Parametrized curves

$\alpha(t) = (a\cos(t), a\sin(t), bt)$

$\alpha(t) = (t^3, t^2)$





Both curves are differentiable, the second curve has $\alpha'(0) = (0,0)$, thus only the first curve is **regular**.

Parametrized surfaces

A parametrized surface element is a regular homeomorphism $\mathbf{x} : U \to S \subset \mathbb{R}^3$. The curves $u \mapsto \mathbf{x}(u, v_0)$ and $v \mapsto \mathbf{x}(u_0, v)$ are the parameter curves of $\mathbf{x}$

Here $U \subset \mathbb{R}^2$ open and $\mathbf{x} : U \to S \subset \mathbb{R}^3$ regular if $\mathrm{d}\mathbf{x}_p : U \to \mathbb{R}^3$ has full rank.

A parametrized surface is the union of parametrized surface elements:

Example of regular surface

Let us show that the unit sphere

$$S^2 = \{(x, y, z) \in \mathbf{R}^3; x^2 + y^2 + z^2 = 1\}$$

is a regular surface.

Consider the parametrization $\mathbf{x}_1 : U \subset \mathbf{R}^2 \to \mathbf{R}^3$ given by

$$\mathbf{x}_1(x, y) = \left(x, y, \sqrt{1-(x^2+y^2)}\right)$$

where $U = \{x, y \in \mathbf{R}^2; x^2 + y^2 < 1\}$.

$\mathbf{x}_1(U)$ is the open part of S^2 above the *xy* plane.

Example of regular surface

Since $x^2 + y^2 < 1$, the function $\sqrt{1-(x^2+y^2)}$ has continuous partial derivatives of all orders and thus $\mathbf{x}_1$ is differentiable.

Similarly, consider the parametrization

$$\mathbf{x}_2(x, y) = \left(x, y, -\sqrt{1-(x^2+y^2)}\right)$$

Observe that $\mathbf{x}_1(U) \cup \mathbf{x}_2(U)$ covers S^2 minus the equator:

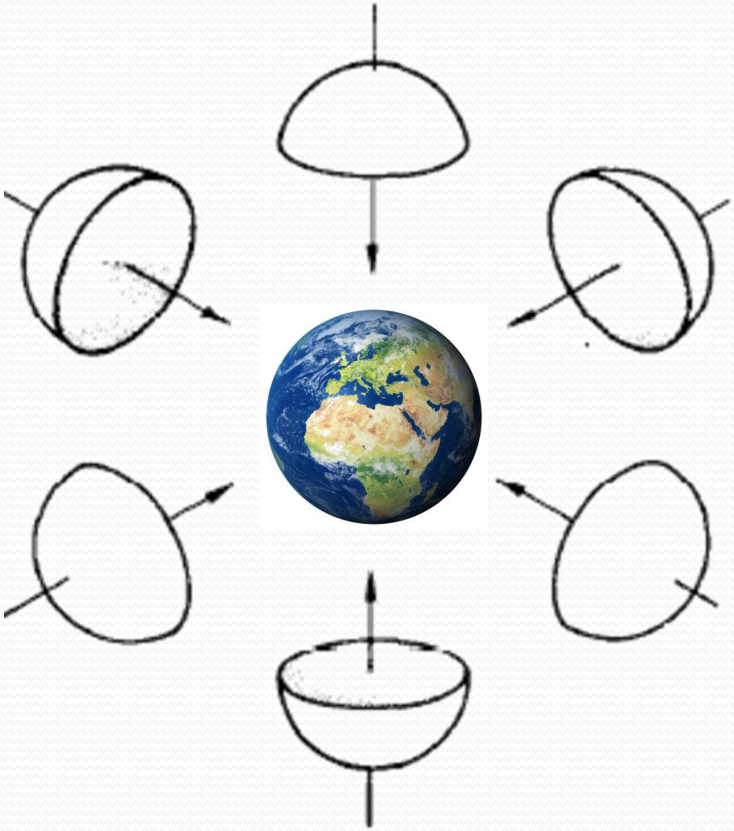
Example of regular surface

We can proceed and define the additional parametrizations:

$$\mathbf{x}_3(x,z) = \left(x, \sqrt{1-(x^2+z^2)}, z\right)$$
$$\mathbf{x}_4(x,z) = \left(x, -\sqrt{1-(x^2+z^2)}, z\right)$$
$$\mathbf{x}_5(y,z) = \left(\sqrt{1-(y^2+z^2)}, y, z\right)$$
$$\mathbf{x}_6(y,z) = \left(-\sqrt{1-(y^2+z^2)}, y, z\right)$$

These, together with $\mathbf{x}_1$ and $\mathbf{x}_2$, cover S^2 completely and show that it is indeed a regular surface.

Example of regular surface



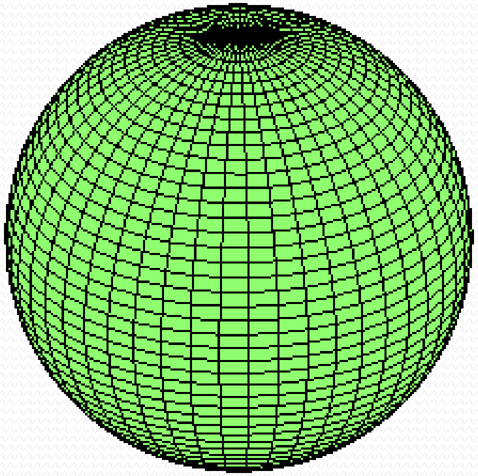
Parametrized surfaces: Examples

$$\mathbf{x} : (-5,5)^2 \to \mathbb{R}^3$$
$$\mathbf{x}(u,v) = (ua\cos(v), ua\sin(v), bv)$$

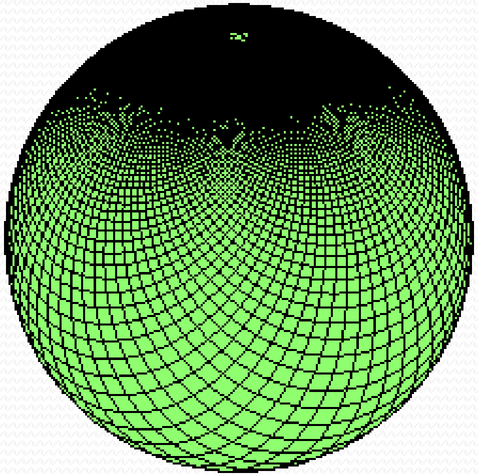
$$\mathbf{x} : (0,6\pi) \times (0,2\pi) \to \mathbb{R}^3$$
$$\mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v))$$
$$x(u,v) = 2(1 - e^{\frac{u}{6\pi}})\cos(u)\cos(\tfrac{v}{2})^2$$
$$y(u,v) = 2(e^{\frac{u}{6\pi}} - 1)\sin(u)\cos(\tfrac{v}{2})^2$$
$$z(u,v) = 1 - e^{\frac{u}{3\pi}} - \sin(v) + e^{\frac{u}{6\pi}}\sin(v)$$

Parametrized surfaces: Examples

$$\mathbf{x} : (0, 2\pi) \times (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \to \mathbb{R}^3$$
$$\mathbf{x}(u, v) = (\cos(u)\cos(v), \sin(u)\cos(v), \sin(v))$$



$$\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3$$
$$\mathbf{x}(u, v) = \tfrac{1}{u^2+v^2+1}(2u, 2v, u^2 + v^2 - 1)$$



Local properties of surfaces

Differential geometry is concerned with those properties of surfaces which depend on their behavior in a neighborhood of a point.

The definition we gave for a *regular surface* seems to be adequate for this purpose. According to this definition, each point of a regular surface belongs to a *surface element* (or *coordinate neighborhood*), and we should be able to define the *local properties* that interest us in terms of these coordinates.

Change of parameters

As also seen from the previous example (the sphere), in general a surface point *can belong to many surface elements*.

In fact, in general we could choose other coordinate systems and parametrizations, for example via stereographic projection or geographical coordinates.

The local properties of the surface should not depend on the specific choice of a system of coordinates!

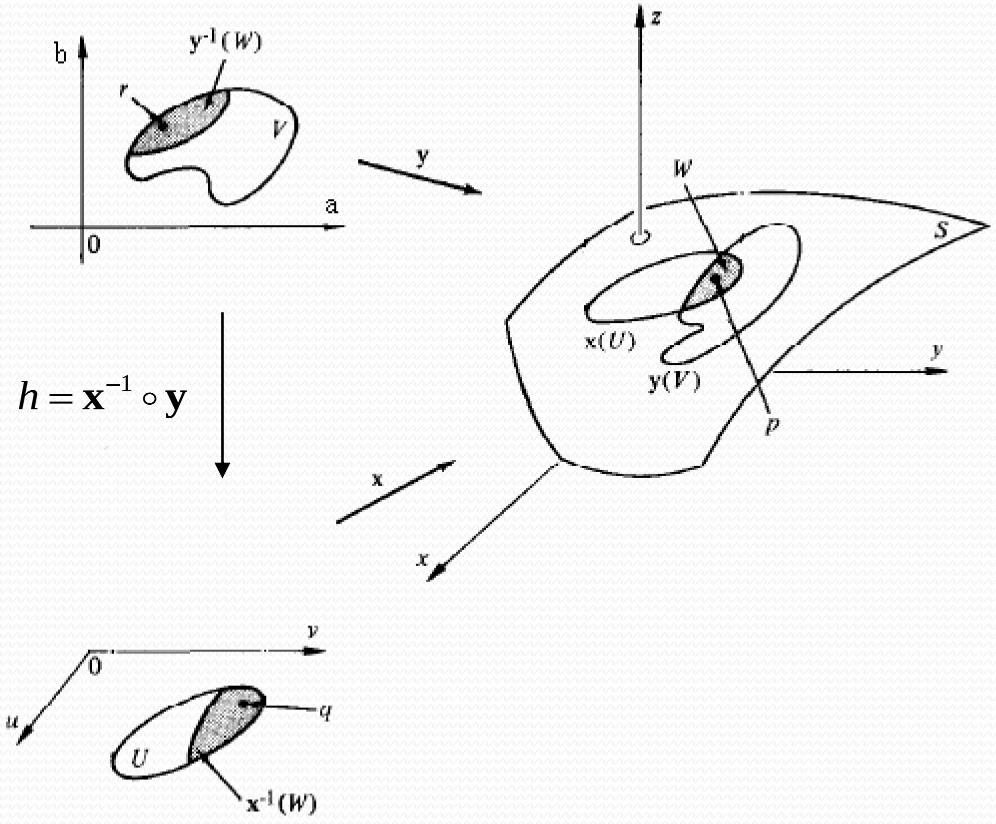
Change of parameters

Fortunately, the following proposition holds:

Let p be a point of a regular surface S, and let $\mathbf{x}: U \subset \mathbf{R}^2 \to S$ and $\mathbf{y}: V \subset \mathbf{R}^2 \to S$ be two parametrizations of S such that $p \in \mathbf{x}(U) \cap \mathbf{y}(V) = W$. Then the *change of coordinates* $h = \mathbf{x}^{-1} \circ \mathbf{y} : \mathbf{y}^{-1}(W) \to \mathbf{x}^{-1}(W)$ is a diffeomorphism (that is, h is differentiable and has a differentiable inverse).

Simply put, if p belongs to two coordinate neighborhoods, with parameters (u,v) and (a,b), it is possible to pass from one of the pairs of coordinates to the other by means of a differentiable transformation.

Change of parameters



Differentiable function on a surface

We will now define the notion of a *differentiable function* on a regular surface.

Let $f : V \subset S \to \mathbf{R}$ be a function defined in an open subset *V* of a regular surface *S*. Then *f* is said to be **differentiable** at *p* if, for some parametrization $\mathbf{x} : U \subset \mathbf{R}^2 \to S$ with $\mathbf{x}(U) \subset V$, the composition $f \circ \mathbf{x} : U \subset \mathbf{R}^2 \to \mathbf{R}$ is differentiable at $\mathbf{x}^{-1}(p)$.

Thus, a function *f* is differentiable at *p* if its expression in the coordinate neighborhood spanned by (*u*,*v*) admits continuous partial derivatives of all orders.

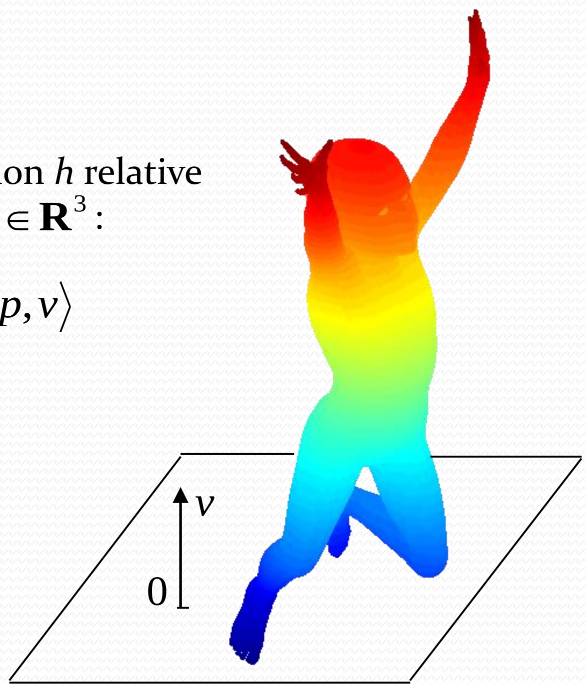
Differentiable function on a surface

Note that this definition *does not depend on the choice of the parametrization*. In fact, if $\mathbf{y}: U \subset \mathbf{R}^2 \to S$ is another parametrization with $p \in \mathbf{x}(V)$, and if $h = \mathbf{x}^{-1} \circ \mathbf{y}$, then $f \circ \mathbf{y} = f \circ \mathbf{x} \circ h$ is also differentiable.

<u>Example:</u>

The height function h relative to a unit vector $v \in \mathbf{R}^3$:

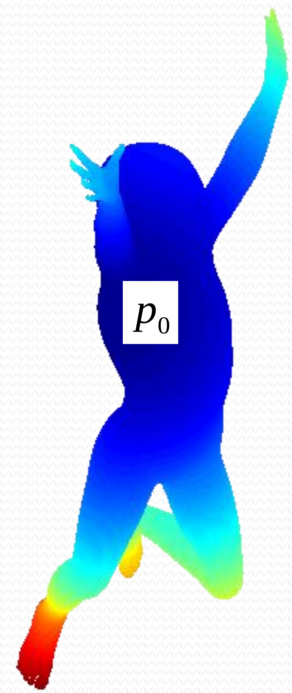
$$h(p) = \langle p, v \rangle$$



<u>Example:</u>

The distance function

$$d(p) = \|p - p_0\|^2$$



Differentiable mappings

We have previously seen the notion of homeomorphic functions among shapes (= continuous in both directions). Similarly, we can extend the definition of differentiability to mappings between surfaces.

A continuous map $\varphi : V_1 \subset S_1 \to S_2$ is said to be differentiable at *p* if, given parametrizations

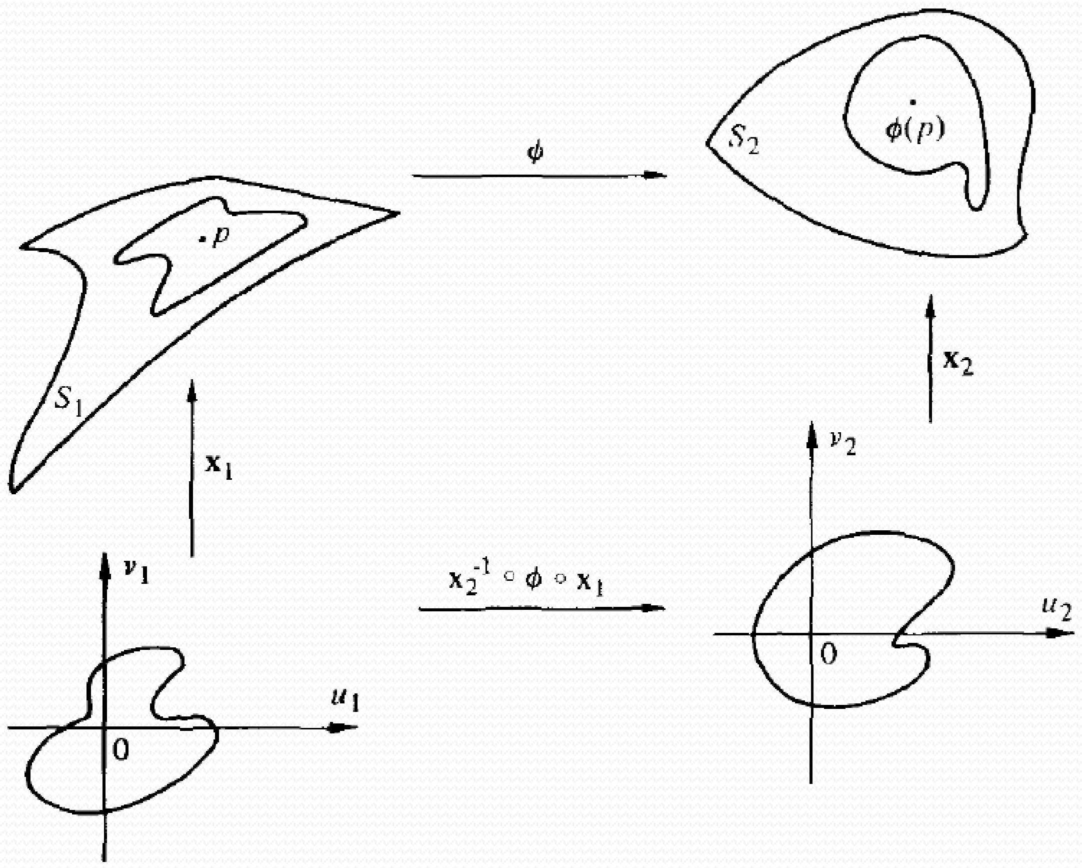
$$\mathbf{x}_1 : U_1 \subset \mathbf{R}^2 \to S_1, \qquad \mathbf{x}_2 : U_2 \subset \mathbf{R}^2 \to S_2$$

with $p \in \mathbf{x}_1(U)$ and $\varphi(\mathbf{x}_1(U_1)) \subset \mathbf{x}_2(U_2)$, the map

$$\mathbf{x}_2^{-1} \circ \varphi \circ \mathbf{x}_1 : U_1 \to U_2$$

is differentiable at $q = \mathbf{x}_1^{-1}(p)$

Differentiable mappings



Diffeomorphisms

We say that two shapes are **diffeomorphic** if there exists a differentiable map between them, with a differentiable inverse. Such a map is called a **diffeomorphism** between the two surfaces.

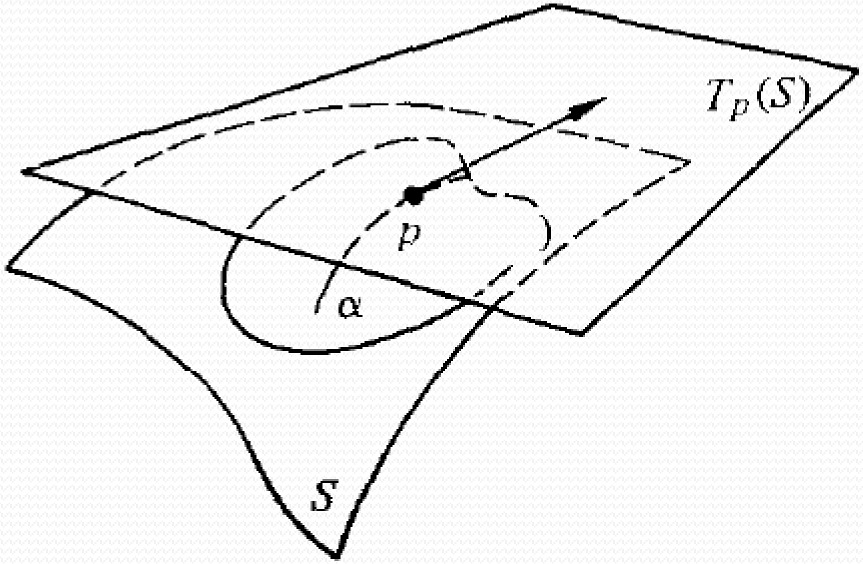
The notion of diffeomorphism plays the same role in the study of regular surfaces that the notion of isometry plays in the study of metric spaces.

From the point of view of differentiability, two diffeomorphic surfaces are indistinguishable.

Also note that every regular surface is locally diffeomorphic to a plane.

Tangent plane

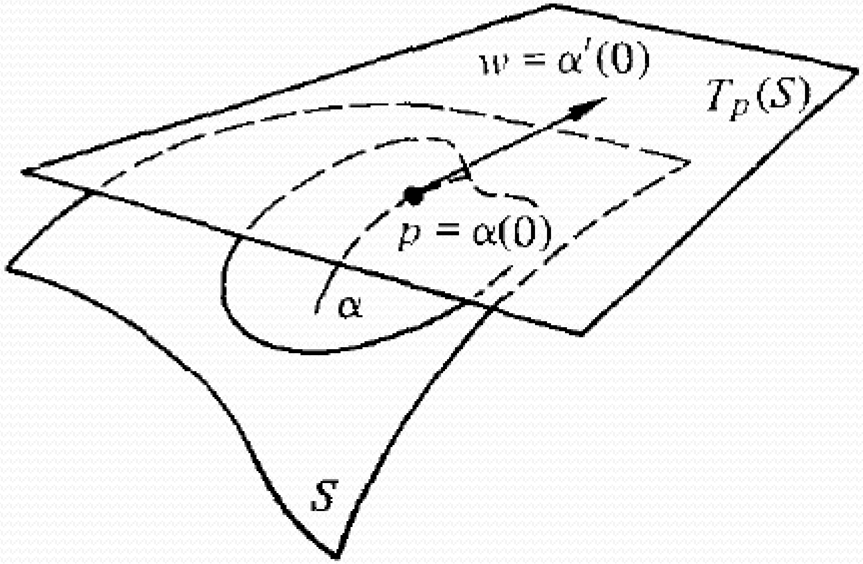
The set of tangent vectors to the parametrized curves of S, passing through p, constitutes the **tangent plane** at p. We will denote it by $T_p(S)$.



Tangent plane

Let us try to be more rigorous. First, note that given a tangent vector $w \in \mathbf{R}^3$ and a point $p_0 \in S \subset \mathbf{R}^3$, we can always find a differentiable curve $\alpha:(-\varepsilon,\varepsilon) \to S$ with $\alpha(0) = p_0$ and $\alpha'(0) = w$.

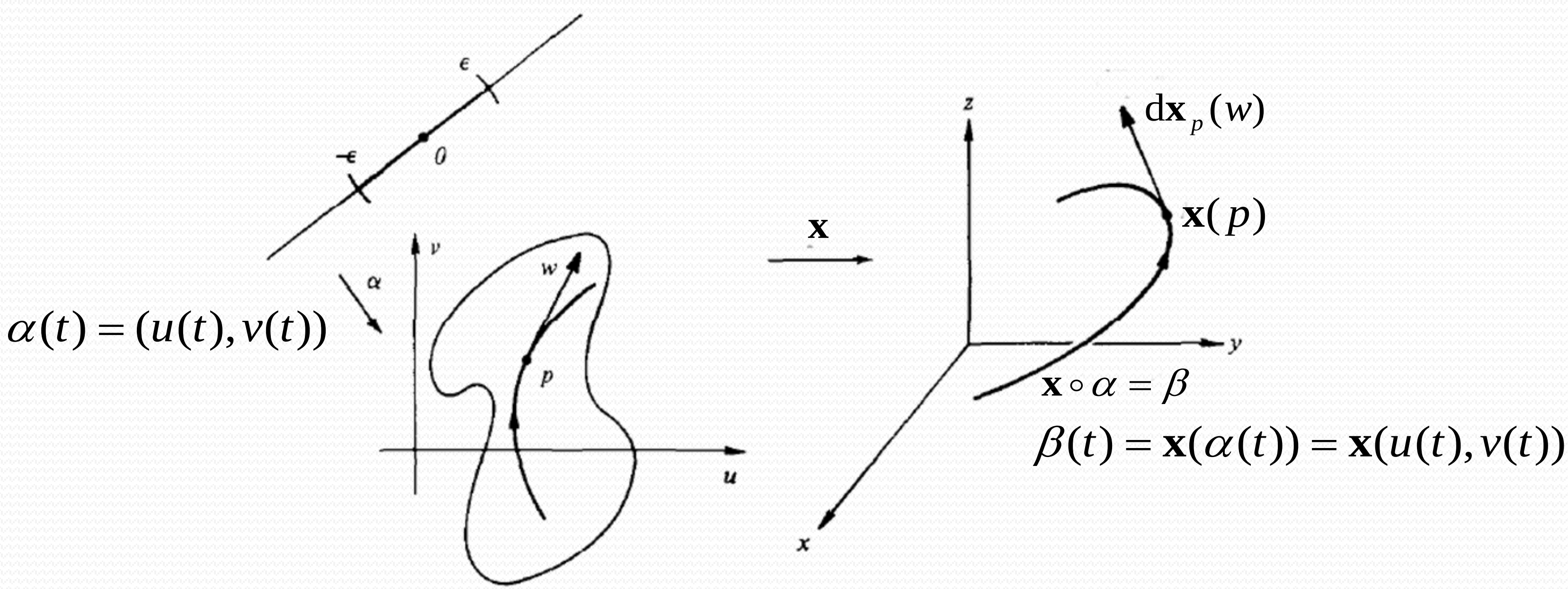
(simply write $\alpha(t) = p_0 + tw$)



Differential of a map

Now let $\mathbf{x}: U \subset \mathbf{R}^2 \to \mathbf{R}^3$ be a differentiable map, and let $\alpha: (-\varepsilon, \varepsilon) \to U$ be a differentiable curve on the parameter domain. Consider the differentiable curve $\beta = \mathbf{x} \circ \alpha : (-\varepsilon, \varepsilon) \to \mathbf{R}^3$. Then the **differential** of **x** at p is defined as:

$$\mathrm{d}\mathbf{x}_p(w) = \beta'(0)$$



Differential of a map

Now let $\mathbf{x}: U \subset \mathbf{R}^2 \to \mathbf{R}^3$ be a differentiable map, and let $\alpha : (-\varepsilon, \varepsilon) \to U$ be a differentiable curve on the parameter domain. Consider the differentiable curve $\beta = \mathbf{x} \circ \alpha : (-\varepsilon, \varepsilon) \to \mathbf{R}^3$. Then the **differential** of **x** at p is defined as:

$$\mathrm{d}\mathbf{x}_p(w) = \beta'(0)$$

• The differential is defined as $\mathrm{d}\mathbf{x}_p : \mathbf{R}^2 \to \mathbf{R}^3$, and is mapping tangent vectors to tangent vectors.

• The differential is a property of **x**, and as such it does *not* depend on the choice of the curve α .

• The differential is a linear map.

The latter two facts are made more evident in the next slide.

Differential of a map

Let (u,v) be coordinates in U and (x,y,z) be coordinates in $\mathbf{R}^3$. Then for the differentiable map $\mathbf{x}: U \subset \mathbf{R}^2 \to \mathbf{R}^3$, we have defined the differential as $\mathrm{d}\mathbf{x}_p(w) = \beta'(0)$, where $\beta(t) = \mathbf{x}(\alpha(t)) = \mathbf{x}(u(t), v(t))$. In order to differentiate β with respect to t, we apply the *chain rule* and obtain, in matrix form:

$$\mathrm{d}\mathbf{x}_p(w) = \underbrace{\begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} \end{pmatrix}}_{\text{"Jacobian matrix" of } \mathbf{x} \text{ at } p} \begin{pmatrix} \dfrac{\partial u}{\partial t} \\ \dfrac{\partial v}{\partial t} \end{pmatrix}$$

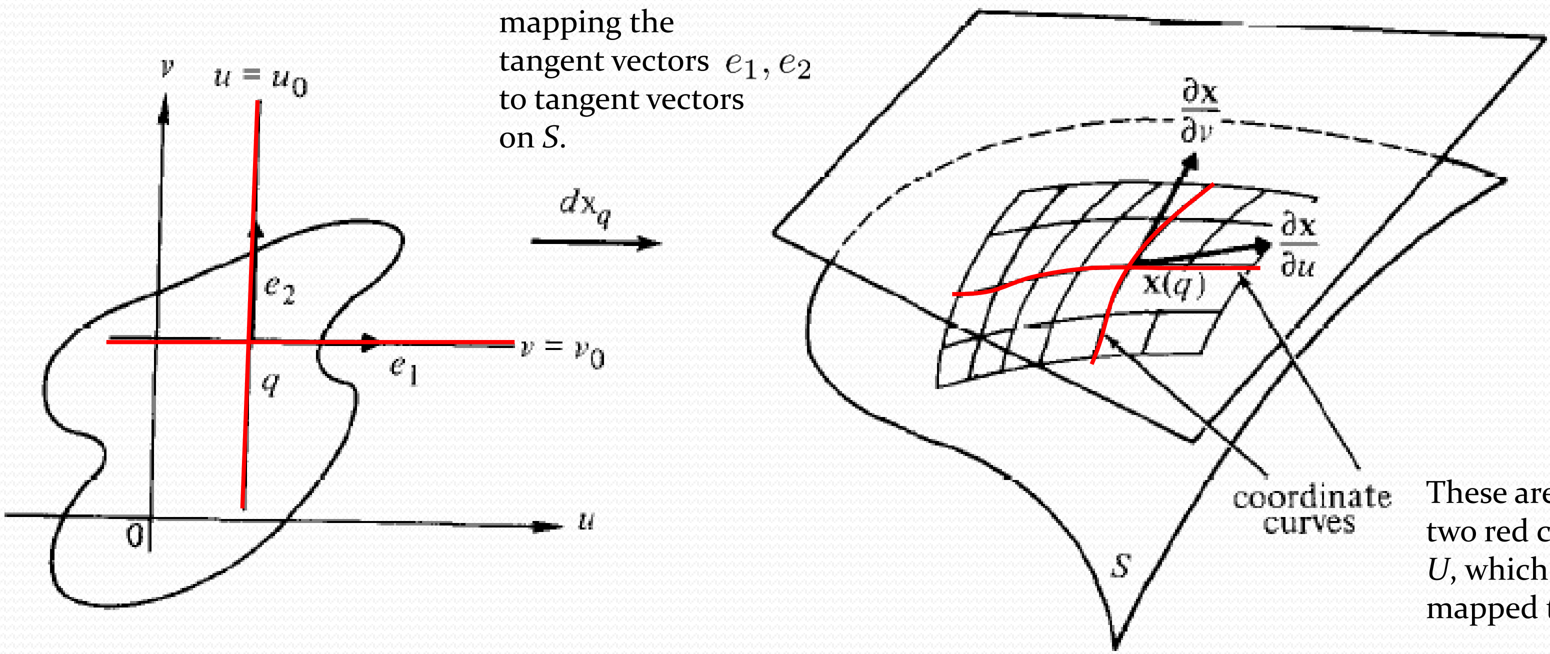
Notice that, indeed, the Jacobian matrix does not depend on the specific curve α that we introduced to define the differential.

Differential of a map

Here $U \subset \mathbb{R}^2$ open and $\mathbf{x} : U \to S \subset \mathbb{R}^3$ regular if $\mathrm{d}\mathbf{x}_p : U \to \mathbb{R}^3$ has full rank.

$$\mathrm{d}\mathbf{x}_q \cdot e_1 = \begin{pmatrix} \vdots & \vdots \\ \frac{\partial \mathbf{x}}{\partial u} & \frac{\partial \mathbf{x}}{\partial v} \\ \vdots & \vdots \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\partial \mathbf{x}}{\partial u}$$

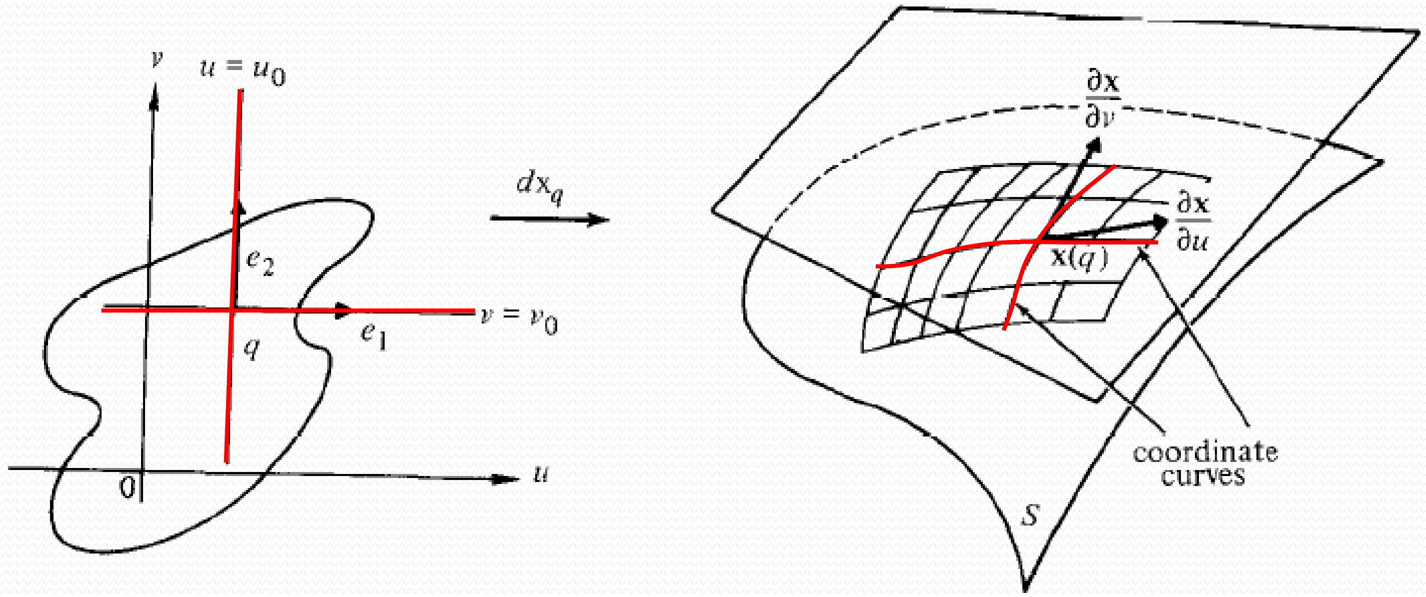
The differential is mapping the tangent vectors e_1, e_2 to tangent vectors on S.



These are just the two red curves in U, which get mapped to S via $\mathbf{x}$.

Tangent plane

We can now give a more rigorous definition for the tangent plane $T_p(S)$.
Let $\mathbf{x}: U \subset \mathbf{R}^2 \to S$ be a parametrization of a regular surface S and let $q \in U$.
The vector subspace of dimension 2, $\mathrm{d}\mathbf{x}_q(\mathbf{R}^2) \subset \mathbf{R}^3$, coincides with the set of tangent vectors to S at $\mathbf{x}(q)$.



First fundamental form

The quadratic form $I_p : T_p(S) \to \mathbf{R}$ given by

$$I_p(w) = \langle w, w \rangle = \|w\|^2$$

is called the **first fundamental form** of the regular surface S at p.

The first fundamental form is, intuitively, the expression of how the surface S "inherits" the natural inner product of $\mathbf{R}^3$.

Geometrically, it allows us to make measurements on the surface (length of curves, areas of regions, etc.) without referring back to the ambient space.

First fundamental form

Let us denote by $\{\mathbf{x}_u, \mathbf{x}_v\}$ the basis associated to a parametrization $\mathbf{x}(u, v)$ at p (thus, $\{\mathbf{x}_u, \mathbf{x}_v\}$ spans the tangent plane $T_p(S)$).

Any vector $w \in T_p(S)$ is the tangent vector to a curve $\alpha(t) = \mathbf{x}(u(t), v(t))$ which lies on the surface, with $t \in (-\varepsilon, \varepsilon)$ and $p = \alpha(0)$.

Then we can write:

$$
\begin{aligned}
I_p(w) = I_p(\alpha'(0)) = \langle \alpha'(0), \alpha'(0) \rangle &\overset{\text{chain rule}}{=} \langle \mathbf{x}_u u' + \mathbf{x}_v v', \mathbf{x}_u u' + \mathbf{x}_v v' \rangle \\
&= \langle \mathbf{x}_u, \mathbf{x}_u \rangle (u')^2 + 2\langle \mathbf{x}_u, \mathbf{x}_v \rangle u'v' + \langle \mathbf{x}_v, \mathbf{x}_v \rangle (v')^2 \\
&= E(u')^2 + 2Fu'v' + G(v')^2
\end{aligned}
$$

First fundamental form

$$I_p(w) = E(u')^2 + 2Fu'v' + G(v')^2$$

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle$$

$$F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle \quad \Longrightarrow \quad I_p(w) = \begin{pmatrix} u' & v' \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}$$

$$G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle$$

E, *F*, and *G* are often called the "coefficients" of the first fundamental form. These coefficients play important roles in many intrinsic properties of the surface.

Note: The first fundamental form is also often called *metric tensor* or *Riemannian metric*. Note however, that this is a different concept than the metric function we introduced in the previous lectures.

First fundamental form

$$I_p(w) = E(u')^2 + 2Fu'v' + G(v')^2$$

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle$$
$$F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle$$
$$G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle$$

$$\Rightarrow \quad I_p(w) = \begin{pmatrix} u' & v' \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}$$

The parametrization **x** is called:

- *orthogonal* if $F \equiv 0$
- *conformal* if, in addition, $E = G$
- *isometric* if, moreover, $E = G \equiv 1$

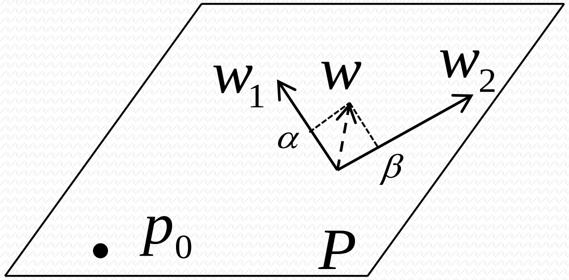
A first (confusing) example

Consider a plane $P \subset \mathbf{R}^3$ passing through p_0 and containing the orthonormal vectors w_1 and w_2.

$$\mathbf{x}(u,v) = p_0 + uw_1 + vw_2$$

We want to compute the first fundamental form for an arbitrary point *w* in *P*. To this end, observe that $\mathbf{x}_u = w_1,\ \mathbf{x}_v = w_2.$

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = 1$$
$$F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0$$
$$G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle = 1$$



Just the usual Pythagorean theorem.

$$I_p(w) = E(u')^2 + 2Fu'v' + G(v')^2 = \alpha^2 + \beta^2$$

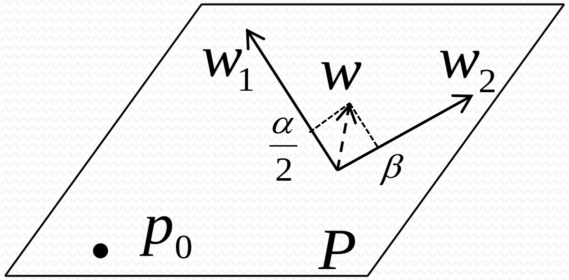
Another (less confusing) example

Consider a plane $P \subset \mathbf{R}^3$ passing through p_0 and containing the orthogonal vectors w_1 and w_2. In this example, $\|w_1\| = 2$ and $\|w_2\| = 1$.

$$\mathbf{x}(u,v) = p_0 + uw_1 + vw_2$$

We want to compute the first fundamental form for an arbitrary point *w* in *P*. To this end, observe that $\mathbf{x}_u = w_1$, $\mathbf{x}_v = w_2$.

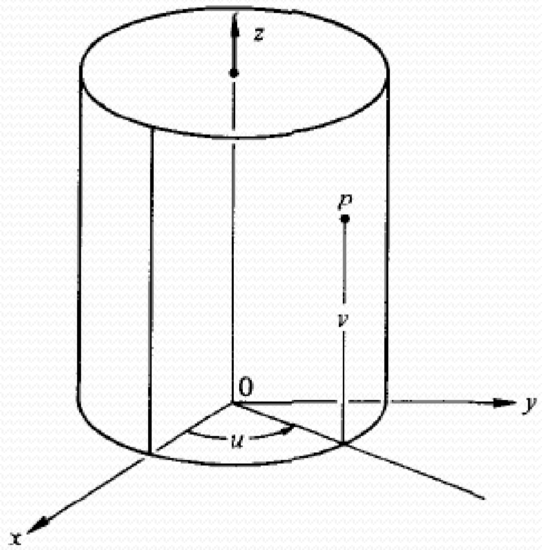
$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = 4$$
$$F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0$$
$$G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle = 1$$



Compared with the previous example, the first fundamental form does not change!

$$I_p(w) = E(u')^2 + 2Fu'v' + G(v')^2 = 4\left(\frac{\alpha}{2}\right)^2 + \beta^2 = \alpha^2 + \beta^2$$

Example



$$\mathbf{x}(u,v) = (\cos u, \sin u, v)$$
$$U = \{(u,v) \in \mathbf{R}^2;\ 0 < u < 2\pi,\ -\infty < v < \infty\}$$

$$\mathbf{x}_u = (-\sin u, \cos u, 0),\ \ \mathbf{x}_v = (0,0,1)$$
$$E = \sin^2 u + \cos^2 u = 1$$
$$F = 0$$
$$G = 1$$

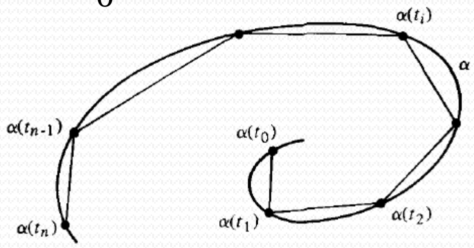
We notice that the plane and the cylinder behave locally in the same way, since their first fundamental forms are equal.

Indeed, plane and cylinder are isometric surfaces!
The first fundamental form is an *isometry invariant*.

Length of a path

By knowing the first fundamental form, we can treat metric questions on a regular surface without further references to the ambient space.

arc-length of a curve $\alpha : (0, T) \to S$ $$s(t) = \int_0^t \|\alpha'(t)\| dt = \int_0^t \sqrt{I(\alpha'(t))} dt$$



where $$\|\alpha'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

Thus, if $\alpha(t) = \mathbf{x}(u(t), v(t))$ is contained in a surface element parametrized by $\mathbf{x}(u,v)$, we can compute the length as:

$$s(t) = \int_0^t \sqrt{E(u')^2 + 2Fu'v' + G(v')^2} dt$$

Area of a region

The first fundamental form can be employed to compute the area of a bounded region R of a regular surface S. If $R \subset S$ is contained in the coordinate neighborhood of the parametrization $\mathbf{x}: U \subset \mathbf{R}^2 \to S$, the **area** of R is defined by

$$A(R) = \iint_Q \|\mathbf{x}_u \times \mathbf{x}_v\| dudv, \qquad Q = \mathbf{x}^{-1}(R)$$

Observing that $\|\mathbf{x}_u \times \mathbf{x}_v\|^2 + \langle \mathbf{x}_u, \mathbf{x}_v \rangle^2 = \|\mathbf{x}_u\|^2 \|\mathbf{x}_v\|^2$, we can rewrite

$$\|\mathbf{x}_u \times \mathbf{x}_v\| = \sqrt{\|\mathbf{x}_u\|^2 \|\mathbf{x}_v\|^2 - \langle \mathbf{x}_u, \mathbf{x}_v \rangle^2} = \sqrt{EG - F^2}$$

Integral of a function

We can follow the same approach to compute the integral of a function defined over the surface, $f : S \to \mathbf{R}$

$$A(R) = \iint_Q f(\mathbf{x}(u,v)) \|\mathbf{x}_u \times \mathbf{x}_v\| du dv, \qquad Q = \mathbf{x}^{-1}(R)$$

Note that we can also write:

$$\|\mathbf{x}_u \times \mathbf{x}_v\| = \sqrt{\left|\det(\mathrm{d}\mathbf{x}_p^{\mathrm{T}} \mathrm{d}\mathbf{x}_p)\right|} \qquad \text{(check it!)}$$

Suggested reading

- *Differential geometry of curves and surfaces*. Do Carmo – Chapters 2.1-2.5, Appendix 2.B
- *Differential Geometry: Curves – Surfaces – Manifolds*. W. Kühnel – Chapter 3A