Analysis of Three-Dimensional Shapes (IN2238, TU München, Summer 2014)

Differential Geometry II
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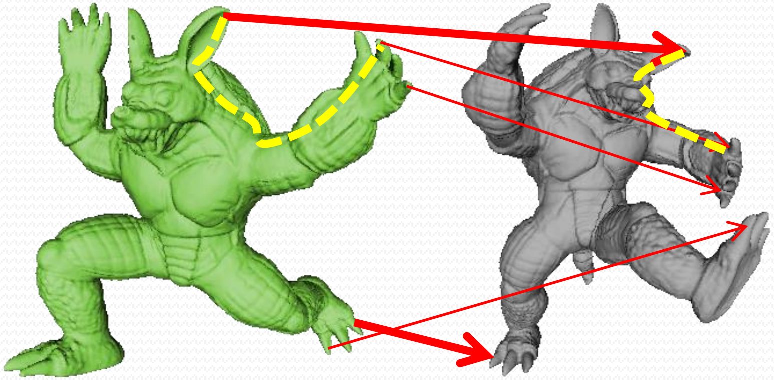
Seminar

«The metric approach to shape matching»

Alfonso Ros

Wednesday, May 28th

14:00 Room 02.09.023



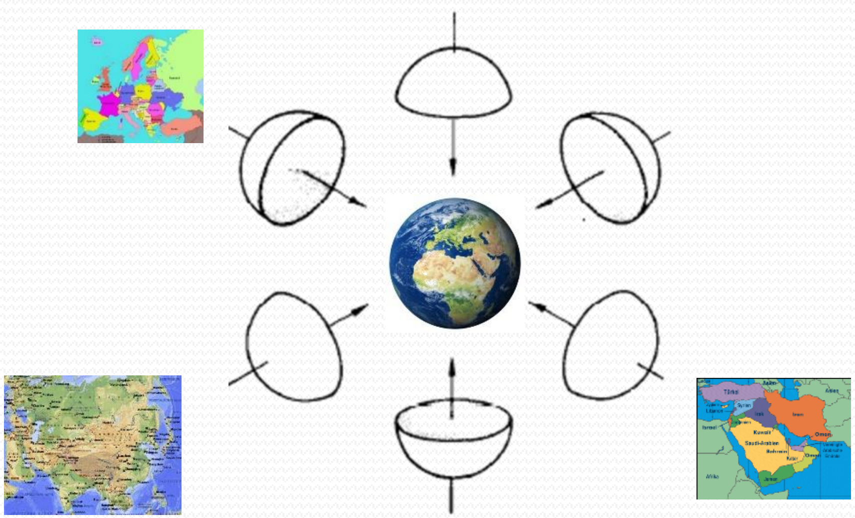
Overview

- Parametrized surfaces and first fundamental form
- Functions defined on surfaces
- Laplace-Beltrami operator
- Extension to triangulated manifolds

Wrap-up

Last time we have introduced the main notions of **differential geometry** that we will be using in this course.

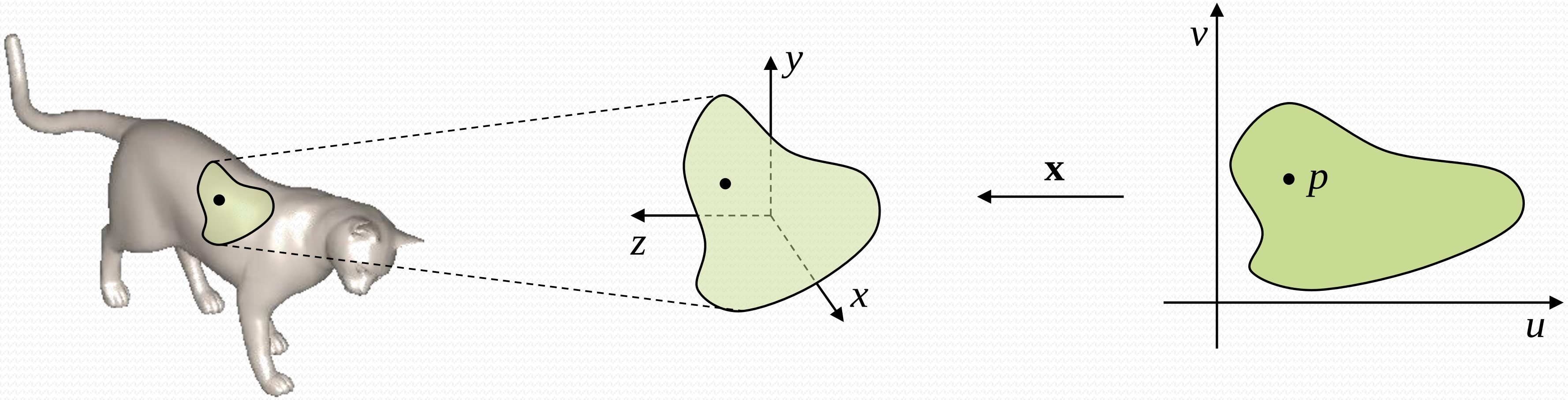
In particular, we showed how to model a 3D shape as a **regular surface**, that is, just a collection of deformed plane patches (called *surface elements*) glued together so as to form something smooth.



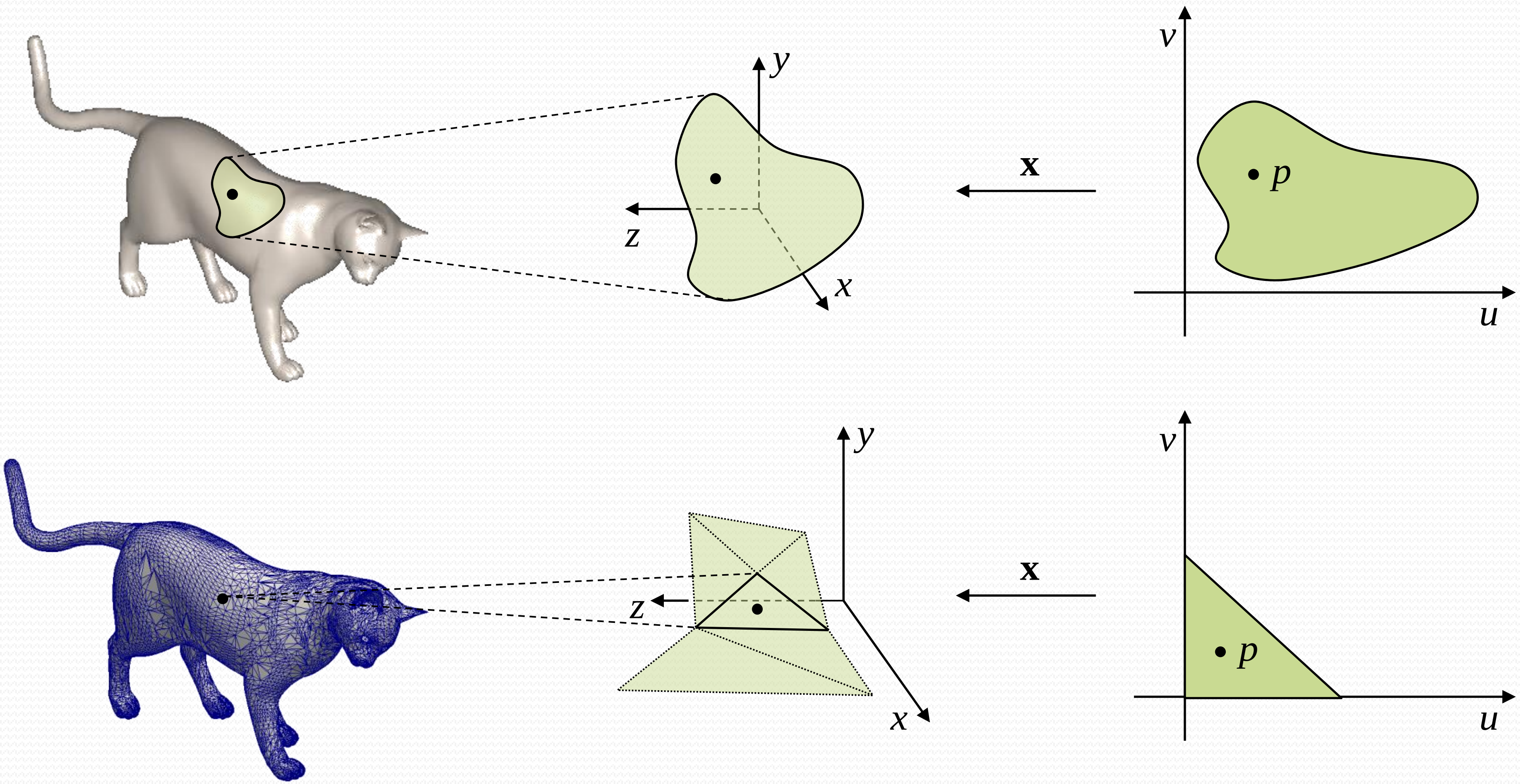
Wrap-up

The general idea of this approach is that we wish to analyze shapes according to a simple recipe:

- Consider each point of the shape as belonging to some surface element.
- Each surface element is the image of a known **parametrization** function **x**.
- Instead of studying the point directly on the surface, we pull it back to the plane and do our calculations there.



Wrap-up

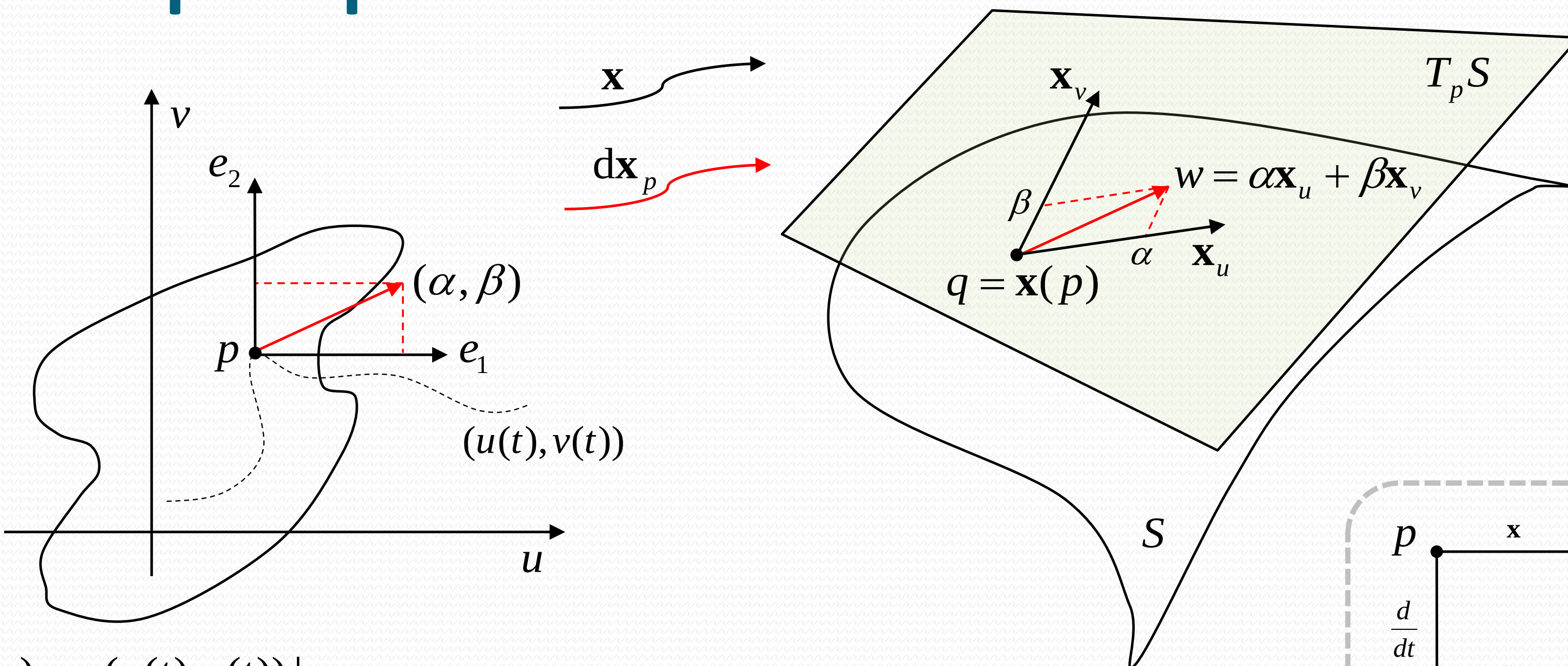


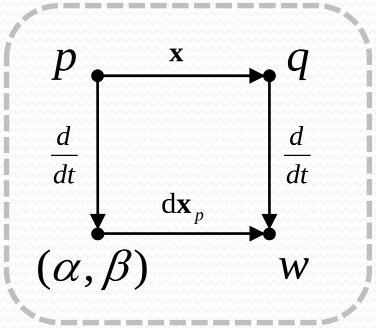
Wrap-up

In doing so, the are some **properties** that we naturally expect to be satisfied:

- The local properties of the surface should not depend on the specific choice of a parametrization **x**.
- Since we want to speak about tangent planes, the parametrization should be *differentiable*.
- Since we know how to do calculus in $\mathbf{R}^n$, we would like to transfer this knowledge to the study of non-Euclidean domains (the surface).

Wrap-up





$$q = \mathbf{x}(p) = \mathbf{x}(u(t), v(t))|_{t=0}$$

$$w = \frac{d}{dt} q \Big|_{t=0} = \frac{d}{dt}\mathbf{x}(u(t), v(t))|_{t=0} = \frac{\partial \mathbf{x}}{\partial u}\frac{du}{dt} + \frac{\partial \mathbf{x}}{\partial v}\frac{dv}{dt} = \mathbf{x}_u u' + \mathbf{x}_v v' = \alpha\mathbf{x}_u + \beta\mathbf{x}_v$$

$$d\mathbf{x}_p((\alpha, \beta)) = w$$

$$d\mathbf{x}_p(e_1) = \mathbf{x}_u$$

$$d\mathbf{x}_p(e_2) = \mathbf{x}_v$$

First fundamental form

We introduced the notion of first fundamental form on a regular surface as the quadratic function $I_p : T_p(S) \to \mathbf{R}$ given by

$$I_p((\alpha, \beta)) = \langle w, w \rangle = \|w\|^2$$

which, given a vector *w* in the tangent plane at *p*, simply computes its length.

In fact, we can generalize this function to take *two* arguments as follows:

$$I_p : T_p(S) \times T_p(S) \to \mathbf{R}$$
$$I_p((\alpha, \beta), (\gamma, \delta)) = \langle (\alpha, \beta), (\gamma, \delta) \rangle$$

The first fundamental form is a tool we use to compute **angles** and **lengths** on the surface (and we actually found out we can also use it to compute **areas**).

First fundamental form

The first fundamental form can be conveniently rewritten as:

$$I_p((\alpha,\beta)) = \langle \alpha\mathbf{x}_u + \beta\mathbf{x}_v, \alpha\mathbf{x}_u + \beta\mathbf{x}_v \rangle = (\alpha \quad \beta)\underbrace{\begin{pmatrix} E & F \\ F & G \end{pmatrix}}_{g}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle \quad F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle \quad G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle$$

«metric tensor»

Or, in case we regard it as the more general bilinear form, as:

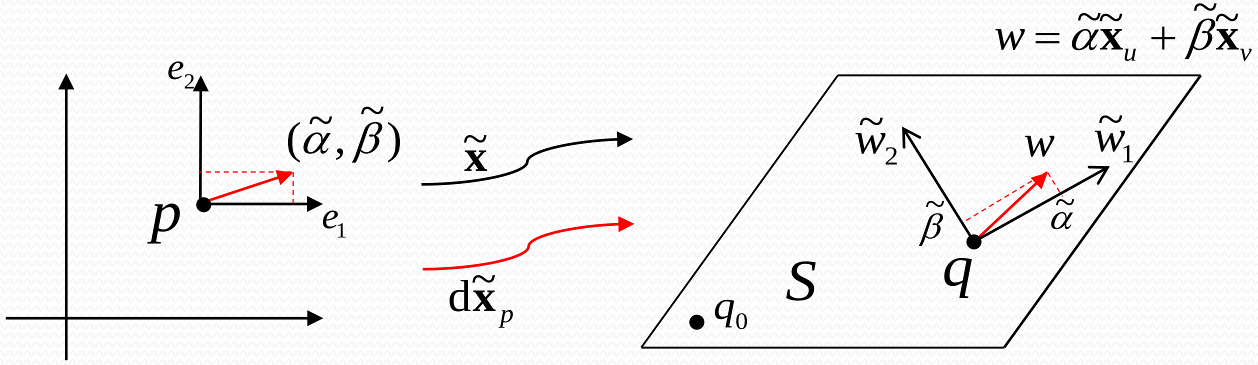
$$I_p((\alpha,\beta),(\gamma,\delta)) = \langle \alpha\mathbf{x}_u + \beta\mathbf{x}_v, \gamma\mathbf{x}_u + \delta\mathbf{x}_v \rangle = (\alpha \quad \beta)\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \gamma \\ \delta \end{pmatrix}$$

The confusing example

Consider a plane $S \subset \mathbf{R}^3$ passing through q_0 and containing the *orthonormal* vectors $\tilde{w}_1$ and $\tilde{w}_2$.

$$\tilde{\mathbf{x}}(u,v) = q_0 + u\tilde{w}_1 + v\tilde{w}_2 \quad \Rightarrow \quad \begin{aligned} \tilde{\mathbf{x}}_u &= \tilde{w}_1 \\ \tilde{\mathbf{x}}_v &= \tilde{w}_2 \end{aligned}$$

We want to compute the first fundamental form for an arbitrary point *q* in *S*.



$$\tilde{g} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Euclidean metric tensor

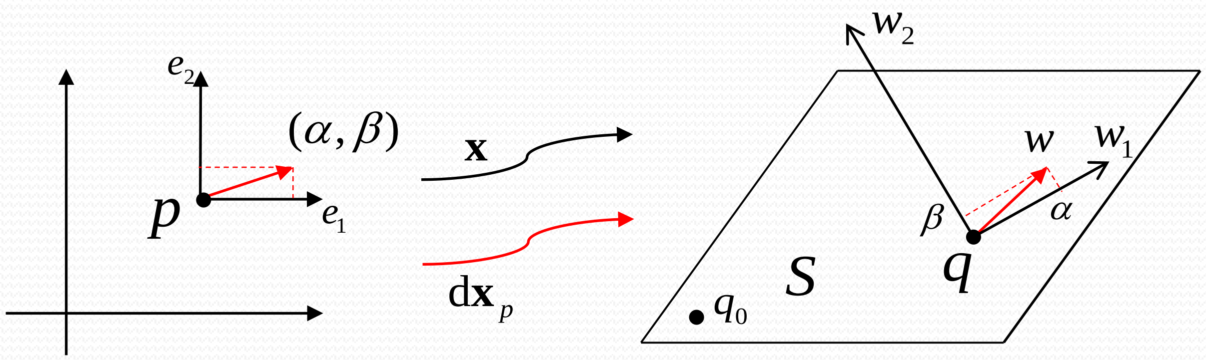
Thus, the first fundamental form of *w* at *p* is $I_p((\tilde{\alpha}, \tilde{\beta})) = \begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} = \tilde{\alpha}^2 + \tilde{\beta}^2$

Example 2 (plane)

Consider the previous example, but this time let $\|w_1\| = 1$ and $\|w_2\| = 2$. We are changing the parametrization **x**, but still we expect that the lengths of vectors in $T_p(S)$ do *not* change (as they are a property of the surface).

Say, for example, that we take the same (p,w) from the previous example.

As before, we have $\mathbf{x}_u = w_1$, $\mathbf{x}_v = w_2$, and then $g = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$.



previous example: $w = \tilde{\alpha}\tilde{\mathbf{x}}_u + \tilde{\beta}\tilde{\mathbf{x}}_v$

this example: $w = \alpha\mathbf{x}_u + \beta\mathbf{x}_v$

The two bases, and thus the coefficients for w are different in the two examples.

$$\alpha\mathbf{x}_u + \beta\mathbf{x}_v \overset{!}{=} \tilde{\alpha}\tilde{\mathbf{x}}_u + \tilde{\beta}\tilde{\mathbf{x}}_v$$

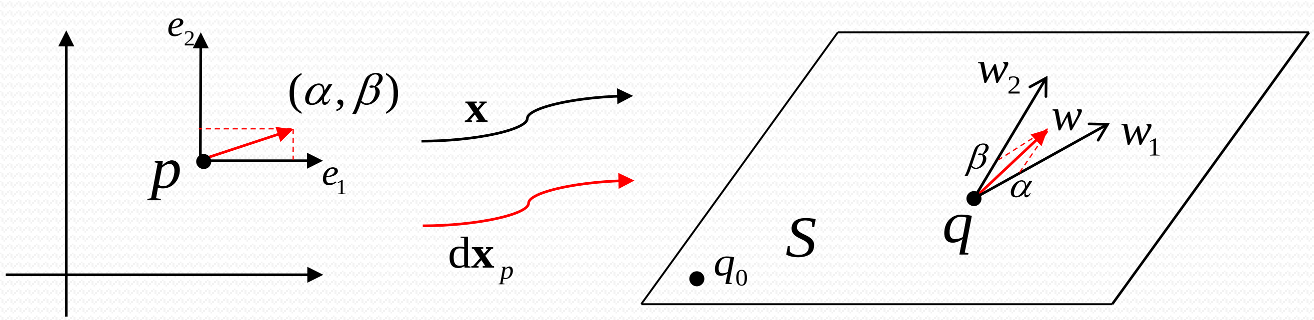
We can now compute $I_p((\alpha,\beta)) = \begin{pmatrix} \alpha & \beta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \tilde{\alpha} & \frac{\tilde{\beta}}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} \tilde{\alpha} \\ \frac{\tilde{\beta}}{2} \end{pmatrix} = \tilde{\alpha}^2 + \tilde{\beta}^2$

Example 3 (plane)

Let's make it more interesting and let $\|w_1\| = 1, \ \ \|w_2\| = 1,$ and $\langle w_1, w_2 \rangle = \dfrac{1}{\sqrt{2}}.$
Again, we expect that the length of *w* does *not* change.

Once again, we have $\mathbf{x}_u = w_1, \ \mathbf{x}_v = w_2$, and now $g = \begin{pmatrix} 1 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1 \end{pmatrix}.$

Even though the metric tensor *g* is different, again we expect the first fundamental form to be the same as before.



$$\alpha w_1 + \beta w_2 \stackrel{!}{=} \tilde{\alpha}\tilde{w}_1 + \tilde{\beta}\tilde{w}_2$$

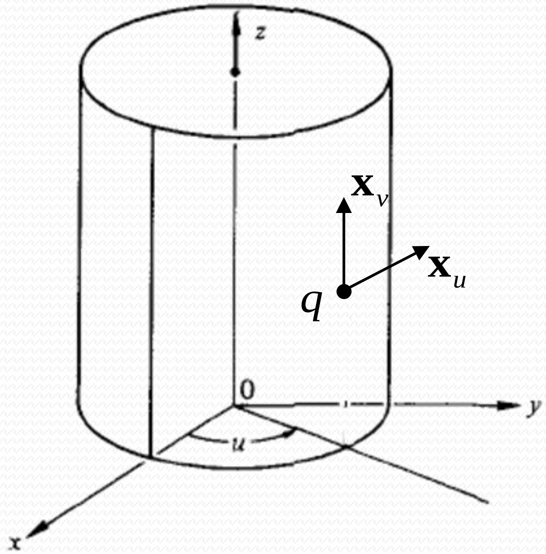
$$\Downarrow$$

$$w_1 = \tilde{w}_1, \quad w_2 = \frac{1}{\sqrt{2}}(\tilde{w}_1 + \tilde{w}_2)$$

$$\alpha = \tilde{\alpha} - \tilde{\beta}, \quad \beta = \sqrt{2}\tilde{\beta}$$

So we get $I_p((\alpha,\beta)) = \begin{pmatrix} \alpha & \beta \end{pmatrix} \begin{pmatrix} 1 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \tilde{\alpha} - \tilde{\beta} & \sqrt{2}\tilde{\beta} \end{pmatrix} \begin{pmatrix} 1 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} \tilde{\alpha} - \tilde{\beta} \\ \sqrt{2}\tilde{\beta} \end{pmatrix} = \tilde{\alpha}^2 + \tilde{\beta}^2$

Example 4 (cylinder)



$$\mathbf{x}(u,v) = (\cos u, \sin u, v)$$

$$U = \{(u,v) \in \mathbf{R}^2;\ 0 < u < 2\pi,\ -\infty < v < \infty\}$$

$$\mathbf{x}_u = (-\sin u, \cos u, 0),\ \ \mathbf{x}_v = (0,0,1)$$

$$E = \sin^2 u + \cos^2 u = 1$$

$$F = 0$$

$$G = 1$$

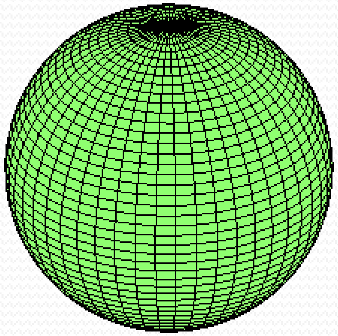
$$\Rightarrow\ g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We notice that the plane and the cylinder behave <u>locally</u> in the same way, since their first fundamental forms are equal.

In other words, plane and cylinder are *locally isometric*. However, the isometry cannot be extended to the entire cylinder because the cylinder is not even homeomorphic to a plane.

Example 5a (sphere)

$$\mathbf{x} : (0, 2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}^3 \qquad \mathbf{x}(u, v) = \begin{pmatrix} \cos(u)\cos(v) \\ \sin(u)\cos(v) \\ \sin(v) \end{pmatrix}$$



$$d\mathbf{x} = \begin{pmatrix} \vdots & \vdots \\ \mathbf{x}_u & \mathbf{x}_v \\ \vdots & \vdots \end{pmatrix} = \begin{pmatrix} -\sin(u)\cos(v) & -\cos(u)\sin(v) \\ \cos(u)\cos(v) & -\sin(u)\sin(v) \\ 0 & \cos(v) \end{pmatrix}$$

$$g = d\mathbf{x}^{\mathrm{T}} d\mathbf{x} = \begin{pmatrix} \cos^2(v) & 0 \\ 0 & 1 \end{pmatrix}$$

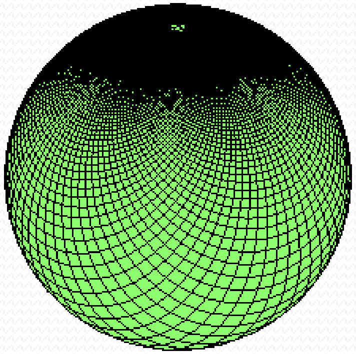
From this example it becomes evident that the coefficients *E*, *F*, *G* are indeed differentiable functions *E*(*u*,*v*), *F*(*u*,*v*), *G*(*u*,*v*).

Thus, if $w = \alpha \mathbf{x}_u + \beta \mathbf{x}_v$ is the tangent vector to the sphere at point **x**(*u*,*v*), then its squared length is given by $|w|^2 = I(w) = \alpha^2 \cos^2(v) + \beta^2$.

Example 5b (sphere)

$$\mathbf{y} : \mathbb{R}^2 \to \mathbb{R}^3 \qquad \mathbf{y}(\tilde{u},\tilde{v}) = \frac{1}{\tilde{u}^2+\tilde{v}^2+1}\begin{pmatrix} 2\tilde{u} \\ 2\tilde{v} \\ \tilde{u}^2+\tilde{v}^2-1 \end{pmatrix}$$

$$d\mathbf{y}(\tilde{u},\tilde{v}) = \frac{2}{(\tilde{u}^2+\tilde{v}^2+1)^2}\begin{pmatrix} \tilde{v}^2-\tilde{u}^2+1 & 2\tilde{u}\tilde{v} \\ 2\tilde{u}\tilde{v} & \tilde{u}^2-\tilde{v}^2+1 \\ 2\tilde{u} & 2\tilde{v} \end{pmatrix}$$



$$g = d\mathbf{y}^{\mathrm{T}} d\mathbf{y}$$

The result is probably going to look not very nice.

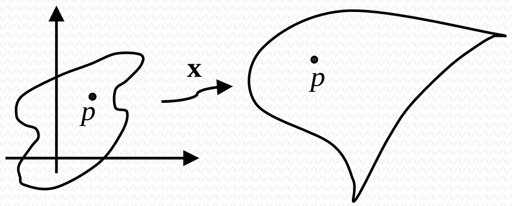
In general, from a computational point of view it is much more convenient to plug in the values for $\tilde{u}, \tilde{v}$ directly in $d\mathbf{y}(\tilde{u},\tilde{v})$, and only then compute g.

Notation

In the following, in order to simplify things we will commit a slight abuse of notation and write:

$$p \equiv \mathbf{x}(p)$$

that is, we identify a point on the surface by its pre-image in the parameter domain



$$w \equiv (\alpha, \beta)$$

in the sense that we identify the vector by its coefficients in the proper basis:

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha b_1 + \beta b_2$$

$$\mathbf{R}^2 : \quad b_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

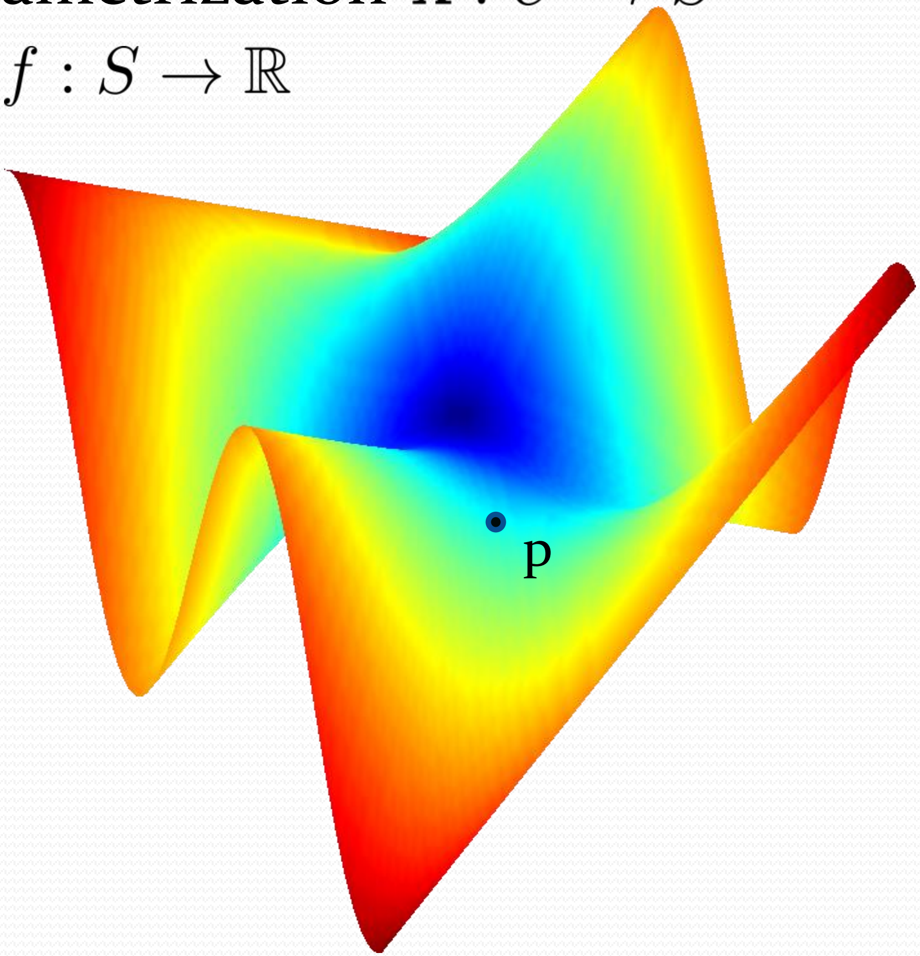
$$\mathbf{R}^3 : \quad b_1 = \mathbf{x}_u, \quad b_2 = \mathbf{x}_v$$

Function on a surface

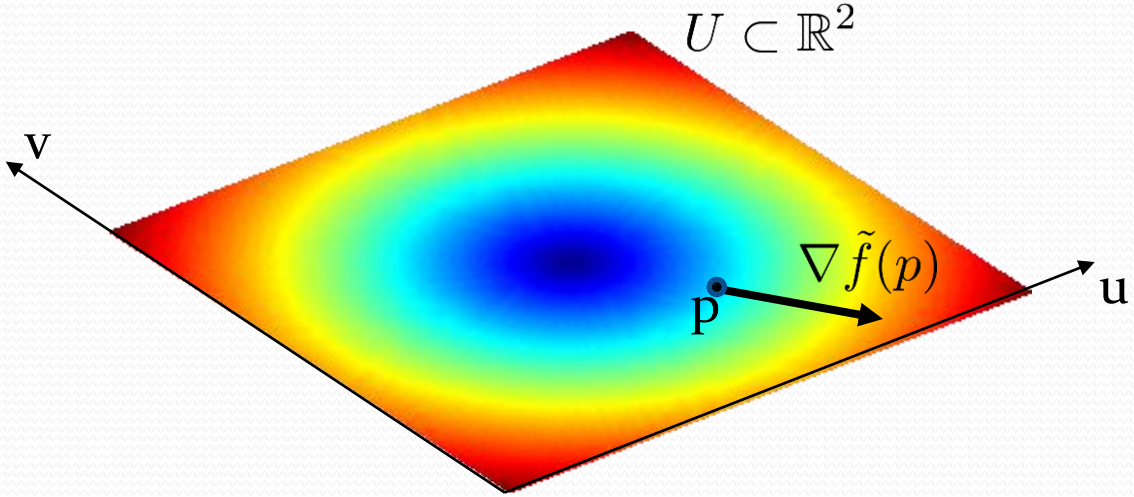
Consider a surface S with parametrization $\mathbf{x} : U \to S$ and a differentiable function $f : S \to \mathbb{R}$

We want to define the gradient $\nabla f(p)$ at a point $p \in S$.

Again this property should be independent of the parametrization



The gradient in R^2

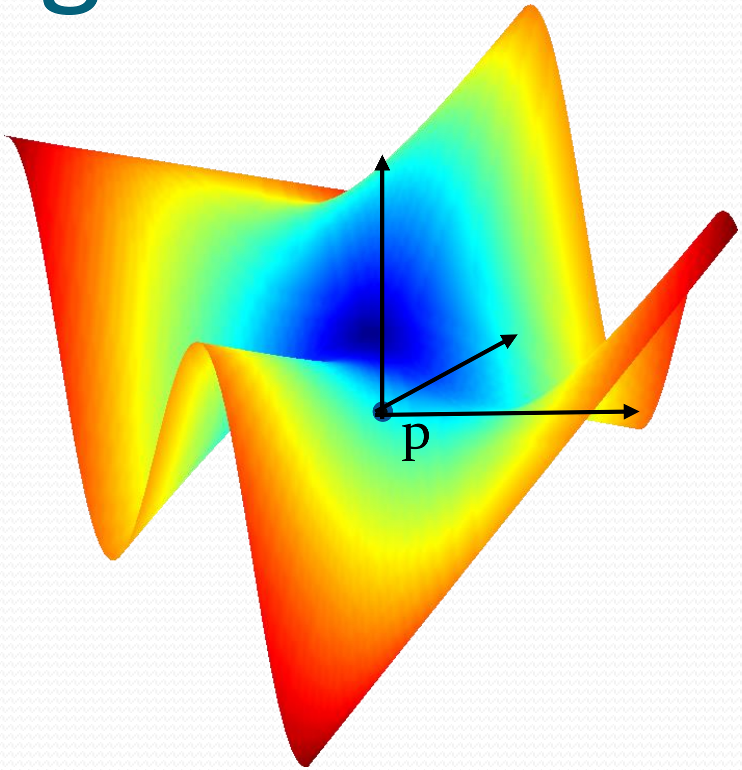


The gradient of a differentiable function $\tilde{f} : U \to \mathbb{R}$ is the vector field

$$\nabla \tilde{f}(p) = \begin{pmatrix} \frac{\partial \tilde{f}}{\partial u}(p) \\ \frac{\partial \tilde{f}}{\partial v}(p) \end{pmatrix}$$

The gradient on a reg. surface

Let now $f : S \to \mathbb{R}$ be a differentiable function.
Ideas how to define $\nabla f(p)$:



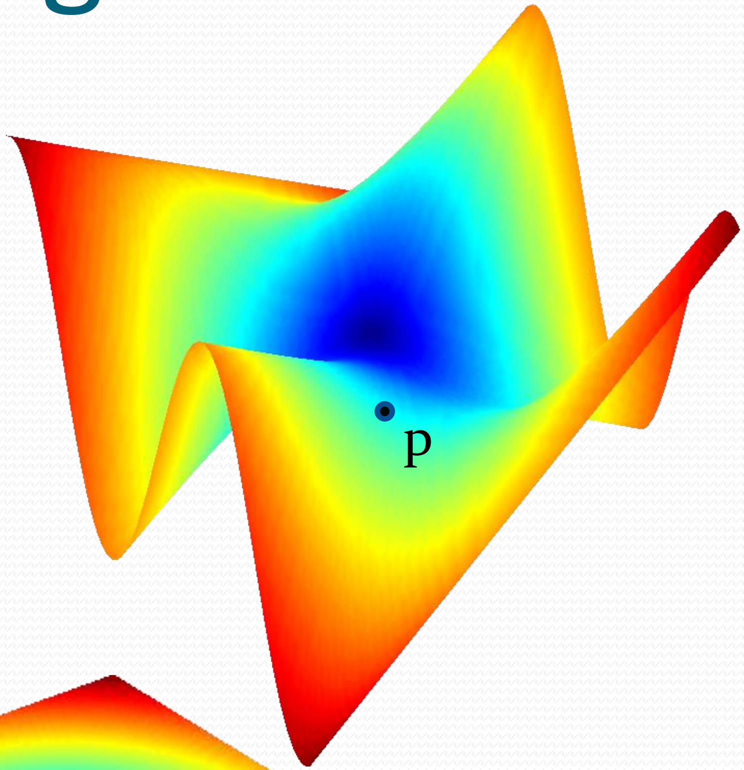
- Use the same formula as before, but in terms of x,y,z:

$$\nabla f(p) = \begin{pmatrix} \frac{\partial f}{\partial x}(p) \\ \frac{\partial f}{\partial y}(p) \\ \frac{\partial f}{\partial z}(p) \end{pmatrix}$$

No information about f outside of S!

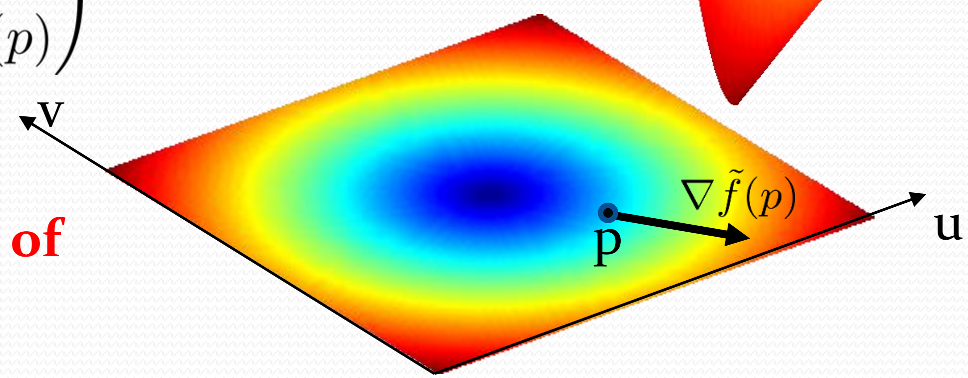
The gradient on a reg. surface

Let now $f : S \to \mathbb{R}$ be a differentiable function.
Ideas how to define $\nabla f(p)$:



- Write f in terms of a parametrization: $\tilde{f}(u_1, u_2) = f(\mathbf{x}(u_1, u_2))$ and set

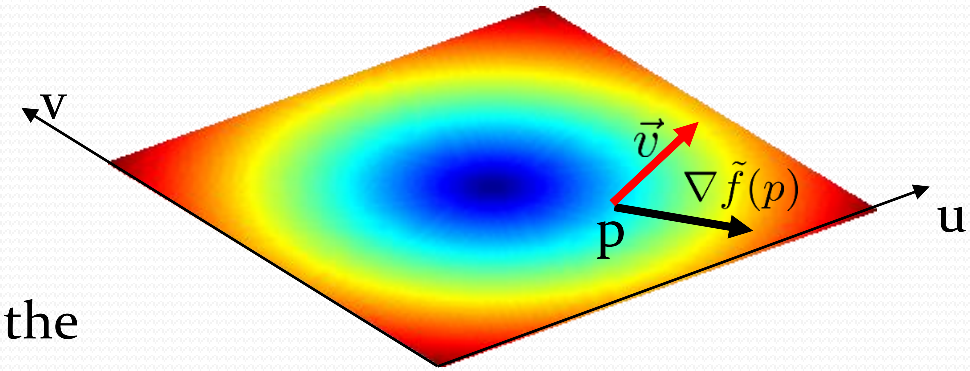
$$\nabla f(p) = \begin{pmatrix} \frac{\partial \tilde{f}}{\partial u}(p) \\ \frac{\partial f}{\partial v}(p) \end{pmatrix}$$



Depends on the choice of the parametrization!

The gradient on a reg. surface

Let now $f : S \rightarrow \mathbb{R}$ be a differentiable function.
Ideas how to define $\nabla f(p)$:



- Interpret the geometric meaning of the gradient
  - the vector that points in the direction of steepest increase of f
  - its length measures the degree of increase
  - directional derivative:

$$\langle \nabla \tilde{f}, \vec{v} \rangle = d\tilde{f}_p(\vec{v}) = \frac{d}{dt}\tilde{f}(p + t\vec{v})|_{t=0}$$

The gradient in on a reg. surface

Definition

The gradient $\nabla f(p) \in T_pS$ is defined via

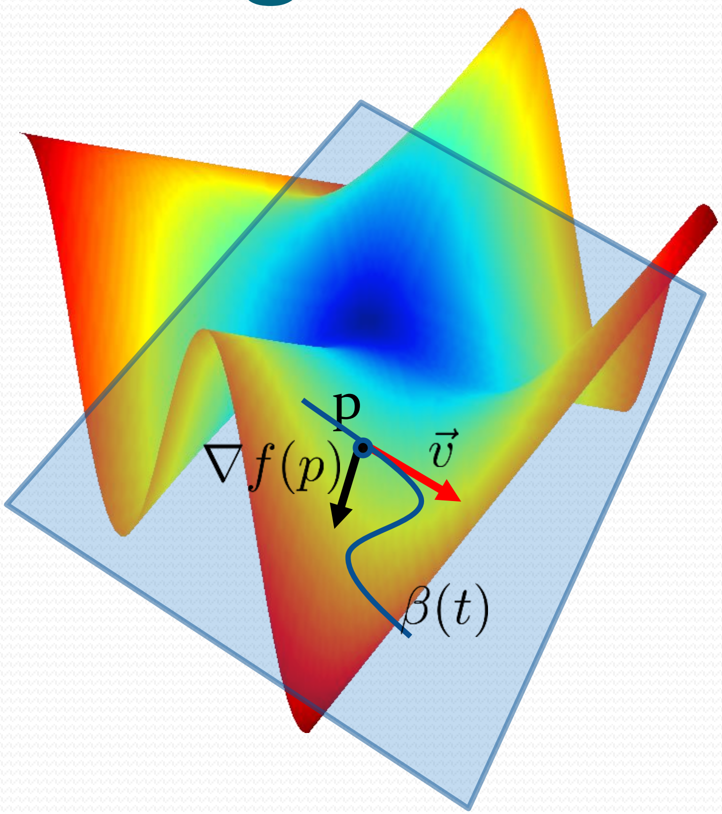
$$I_p(\nabla f, \vec{v}) = df_p(\vec{v})$$

$$= \frac{d}{dt} f(\beta(t))|_{t=0} \quad \forall \vec{v} \in T_pS$$

where

$$\beta(0) = p$$

$$\dot{\beta}(0) = \vec{v}$$

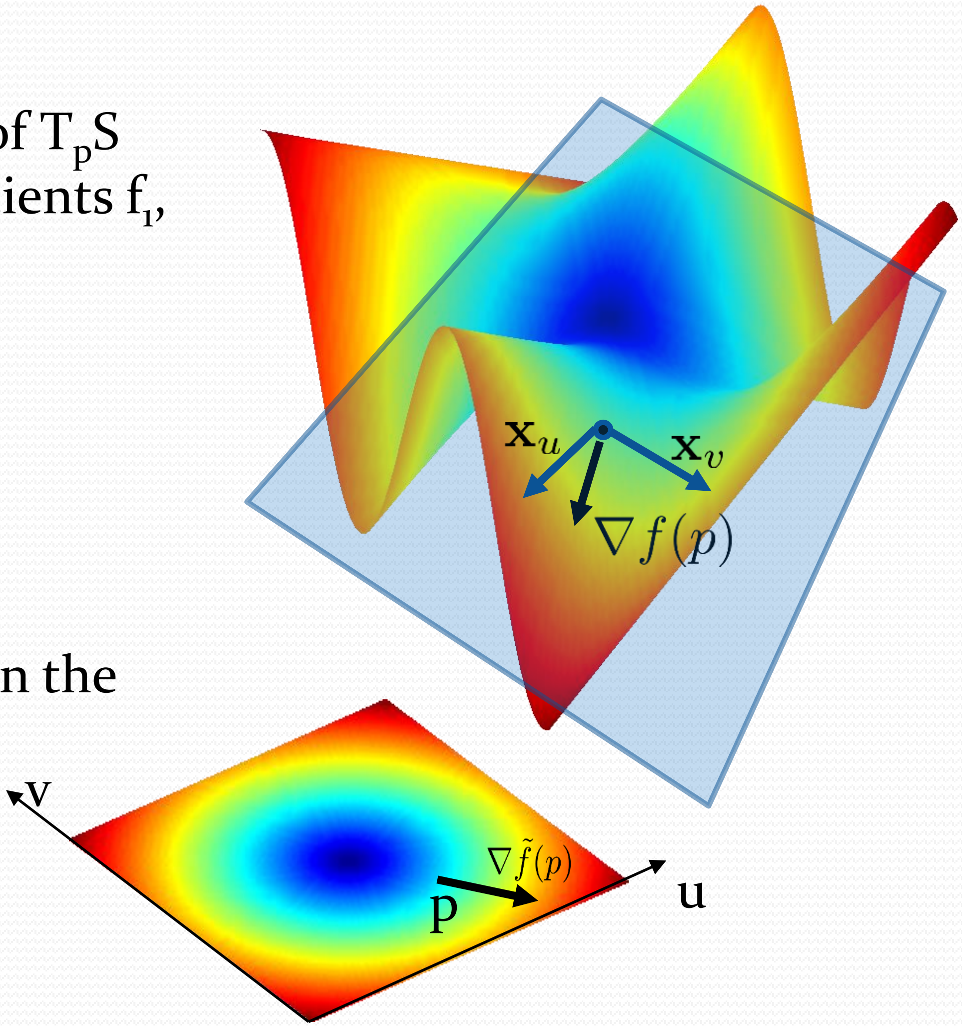


The gradient in local coordinates

Since the gradient is a member of T_pS we should be able to find coefficients f_1, f_2 such that

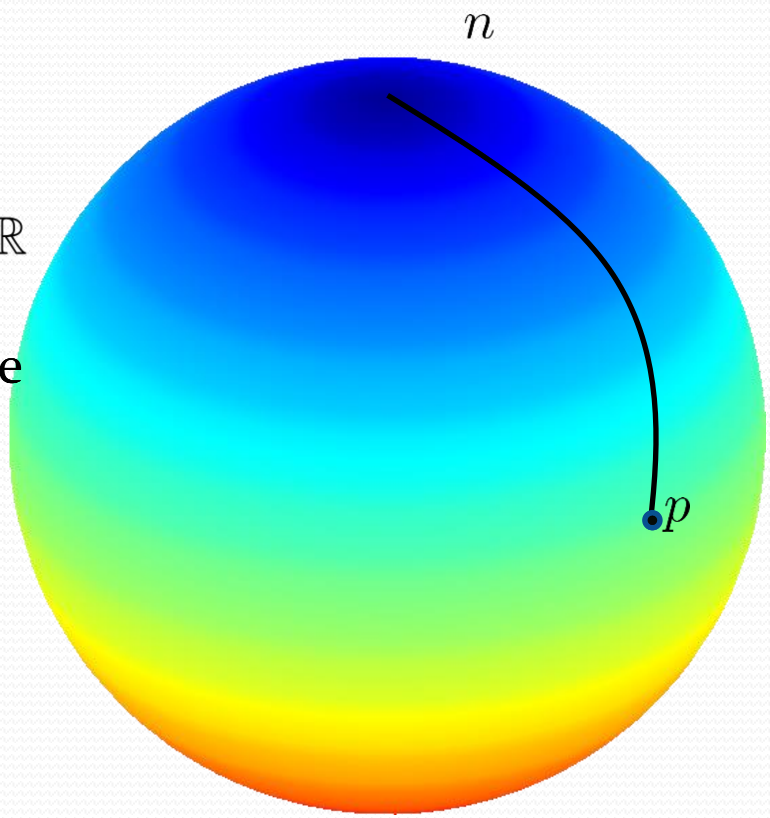
$$\nabla f(p) = f_1 \mathbf{x}_u + f_2 \mathbf{x}_v$$
$$= d\mathbf{x} \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

These coefficients will depend on the gradient of $\tilde{f} = f \circ \mathbf{x} : U \to \mathbb{R}$

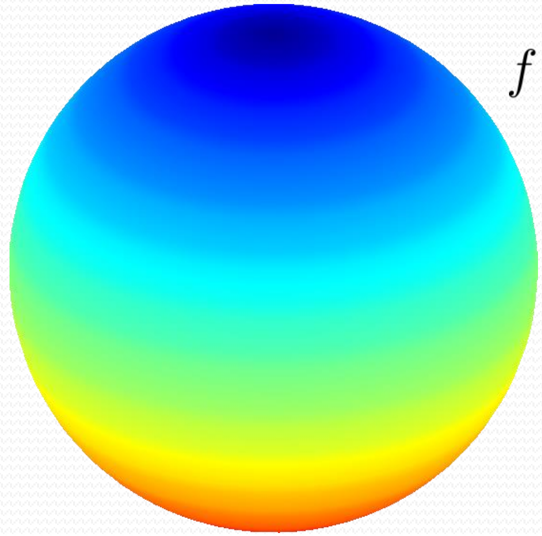


An example

Consider the function $f : \mathbb{S}^2 \setminus \{n\} \to \mathbb{R}$ that assigns to each point on the unitsphere its distance to the north pole n: $f(p) = d_{\mathbb{S}^2}(n, p)$



An example



$$f(p) = d_{\mathbb{S}^2}(n,p)$$

$$\mathbf{x} : (0, 2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}^3$$

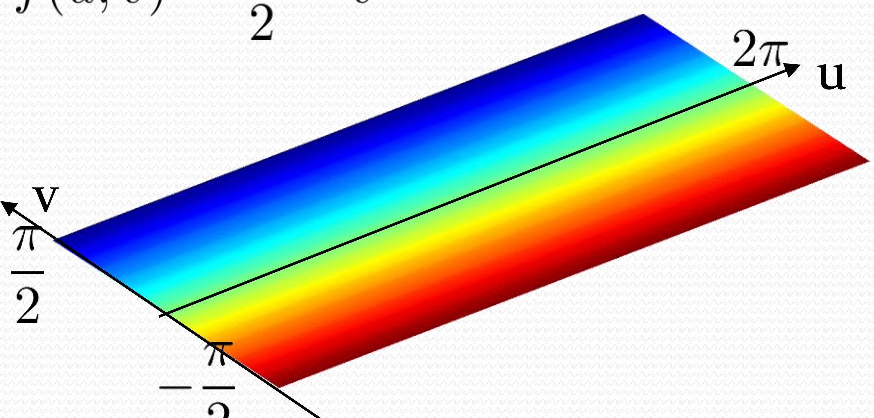
$$\mathbf{x}(u,v) = \begin{pmatrix} \cos(u)\cos(v) \\ \sin(u)\cos(v) \\ \sin(v) \end{pmatrix}$$

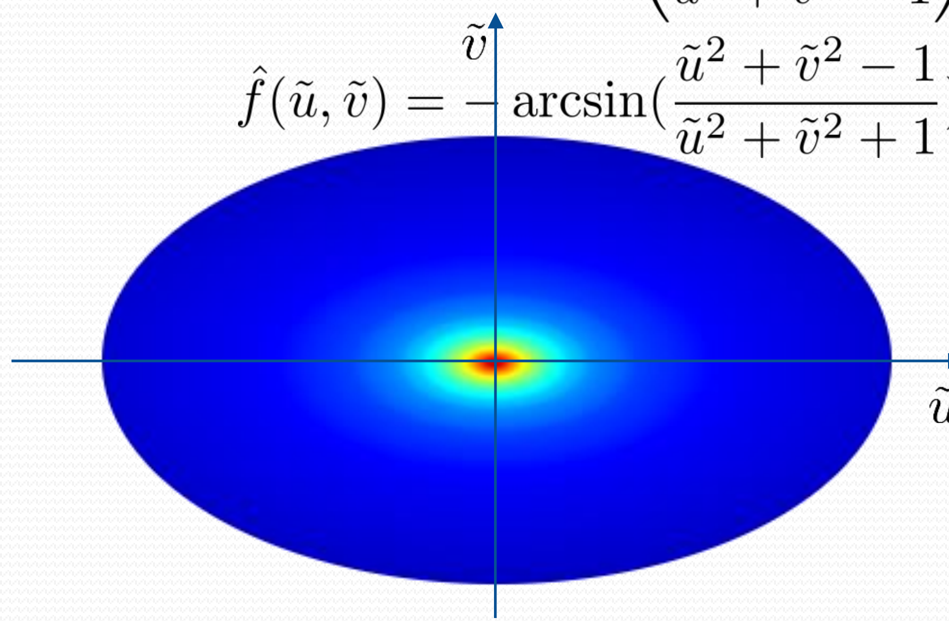
$$\mathbf{y} : \mathbb{R}^2 \to \mathbb{R}^3$$

$$\mathbf{y}(\tilde{u},\tilde{v}) = \frac{1}{\tilde{u}^2 + \tilde{v}^2 + 1} \begin{pmatrix} 2\tilde{u} \\ 2\tilde{v} \\ \tilde{u}^2 + \tilde{v}^2 - 1 \end{pmatrix}$$

$$\hat{f}(\tilde{u},\tilde{v}) = -\arcsin(\frac{\tilde{u}^2 + \tilde{v}^2 - 1}{\tilde{u}^2 + \tilde{v}^2 + 1})$$

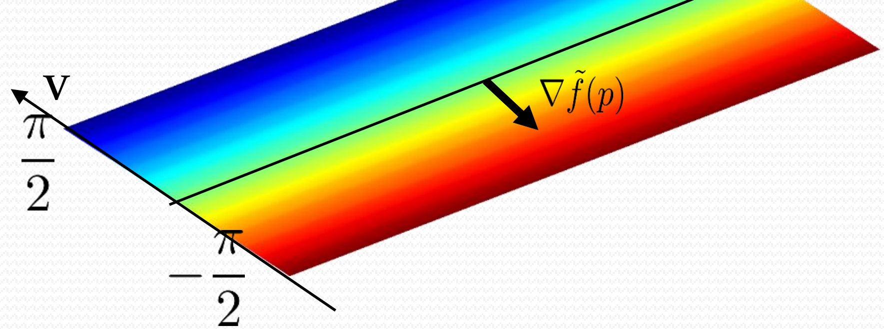
$$\tilde{f}(u,v) = \frac{\pi}{2} - v$$



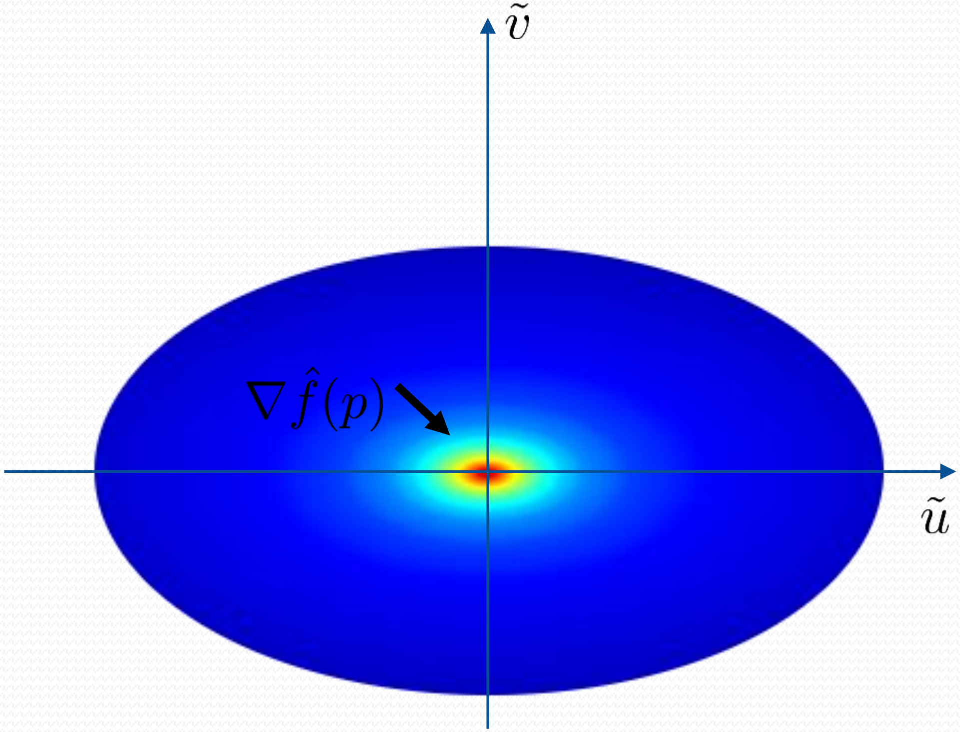


An example

$$\tilde{f}(u,v) = \frac{\pi}{2} - v$$



$$\nabla \tilde{f}(u,v) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

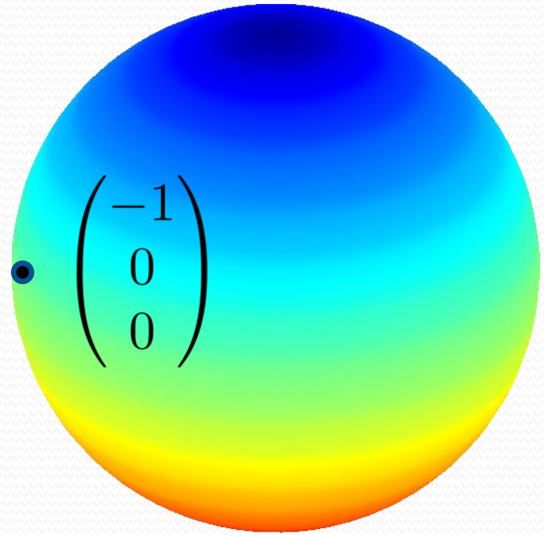


$$\hat{f}(\tilde{u},\tilde{v}) = -\arcsin\left(\frac{\tilde{u}^2+\tilde{v}^2-1}{\tilde{u}^2+\tilde{v}^2+1}\right)$$

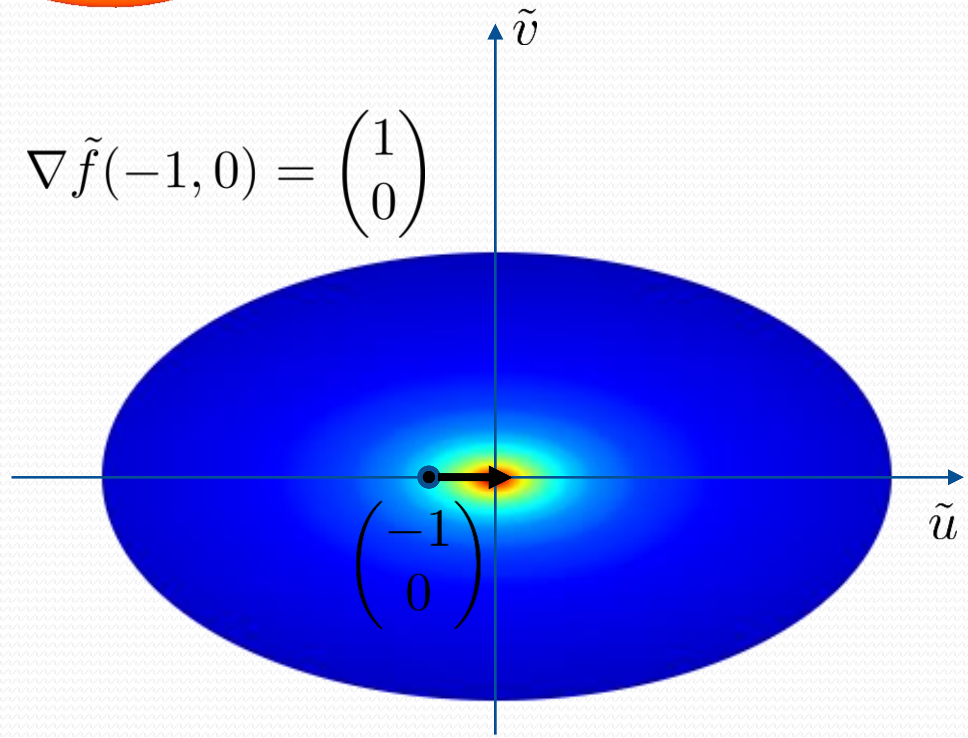
$$\nabla \hat{f}(\tilde{u},\tilde{v}) = \frac{-2}{r(r^2+1)} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}$$

An example

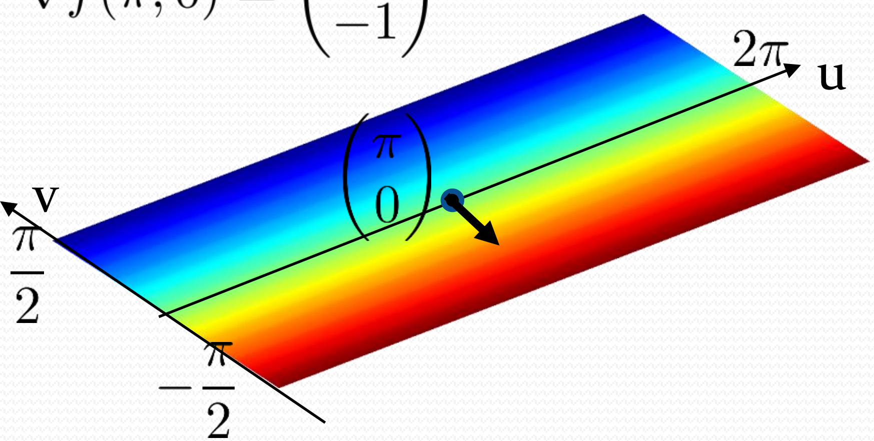
$$\nabla f(-1,0,0) = ?$$



$$\nabla \tilde{f}(-1,0) = \begin{pmatrix}1\\0\end{pmatrix}$$



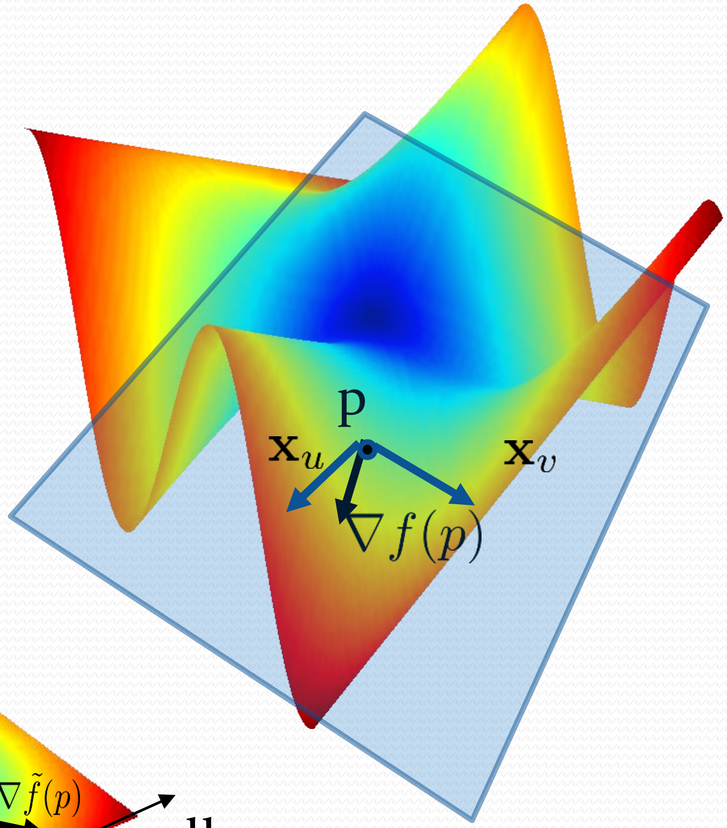
$$\nabla \tilde{f}(\pi,0) = \begin{pmatrix}0\\-1\end{pmatrix}$$



The gradient in local coordinates

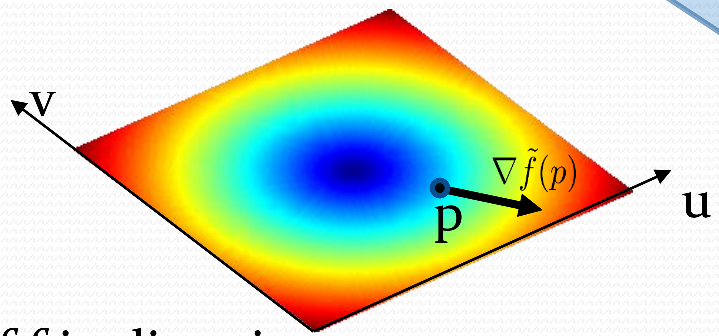
Aim:
calculate the coefficients of

$$\nabla f(p) = f_1 \mathbf{x}_u + f_2 \mathbf{x}_v = d\mathbf{x} \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$



Let us first write

$$\frac{\partial \tilde{f}}{\partial u} = \frac{\partial (f \circ \mathbf{x})}{\partial u} = df\left(\frac{\partial \mathbf{x}}{\partial u}\right)$$



which is the change of f in direction $\mathbf{x}_u$

The gradient in local coordinates

Let now $\vec{v} = v_1\mathbf{x}_u + v_2\mathbf{x}_v$

The linearity of df_p yields

Exercise

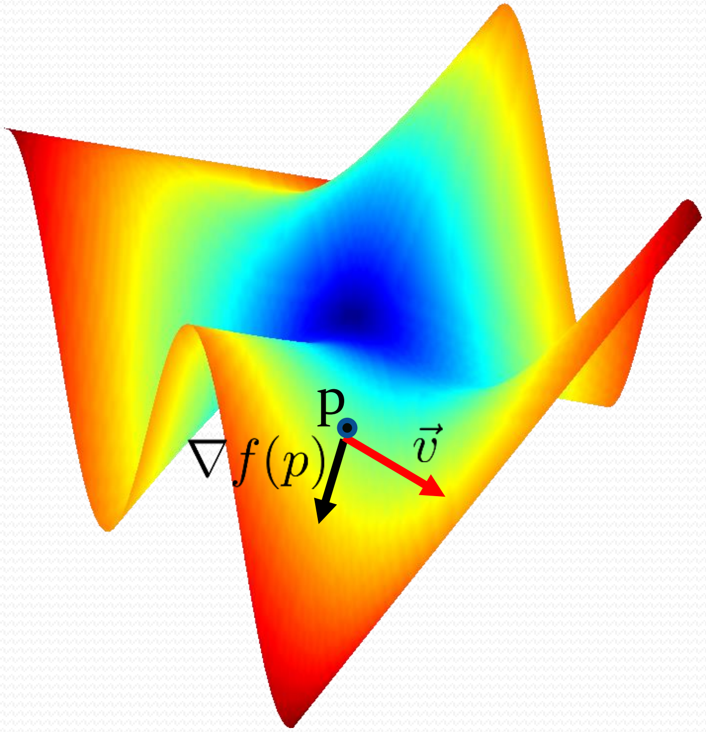
$$df_p(\vec{v}) = v_1 df_p(\mathbf{x}_u) + v_2 df_p(\mathbf{x}_v)$$
$$= v_1 \frac{\partial \tilde{f}}{\partial u} + v_2 \frac{\partial \tilde{f}}{\partial v}$$
$$= (\nabla \tilde{f})^T \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

On the other hand

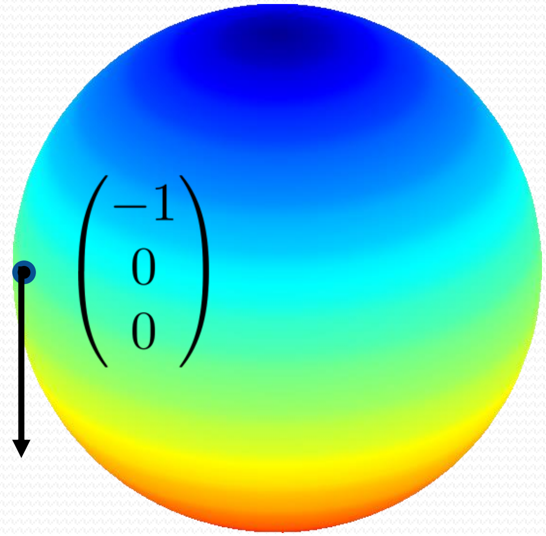
$$df_p(\vec{v}) = I_p(\nabla f, \vec{v}) = \begin{pmatrix} f_1 & f_2 \end{pmatrix} g \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

This means

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = g^{-1} \nabla \tilde{f}$$



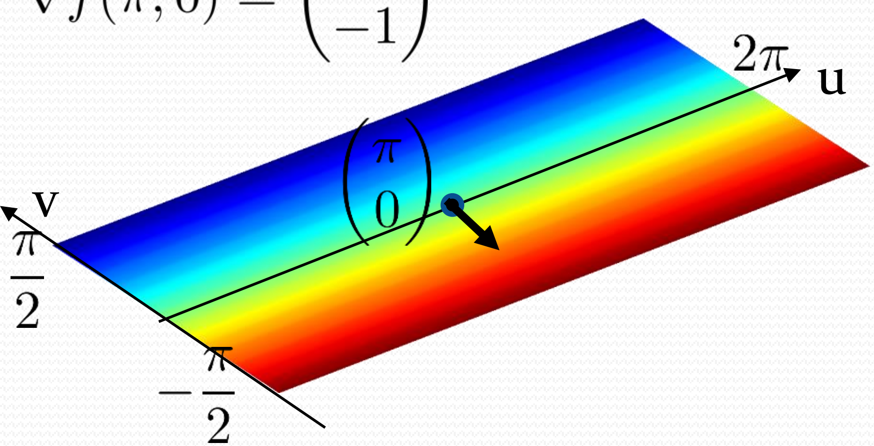
First parametrization



$$d\mathbf{x}(\pi, 0) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$g_{\mathbf{x}}(\pi, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (g_{\mathbf{x}}(\pi, 0))^{-1}$$

$$\nabla \tilde{f}(\pi, 0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

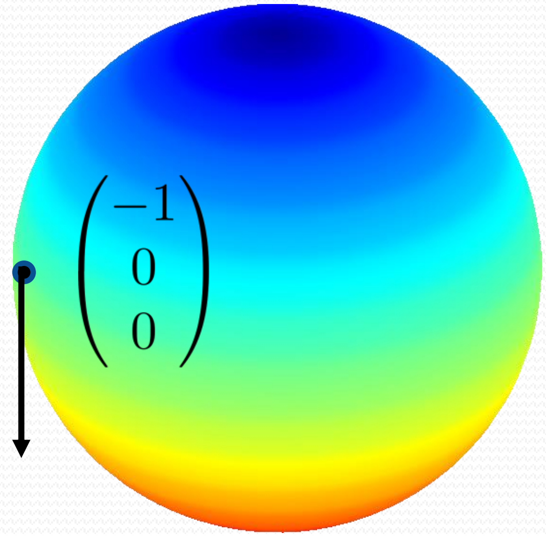


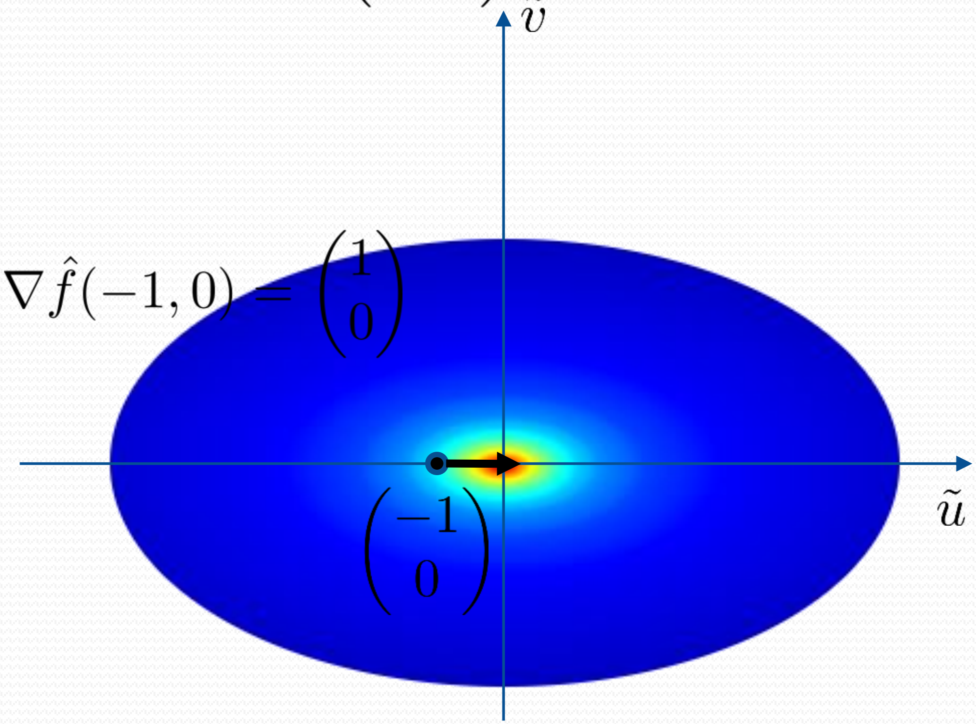
$$\begin{aligned} \nabla f(-1, 0, 0) &= d\mathbf{x} \cdot g_{\mathbf{x}}^{-1} \cdot \nabla \tilde{f}(\pi, 0) \\ &= d\mathbf{x} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \end{aligned}$$

Second parametrization

$$d\mathbf{y}(-1,0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$g_{\mathbf{y}}(-1,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (g_{\mathbf{y}}(-1,0))^{-1}$$





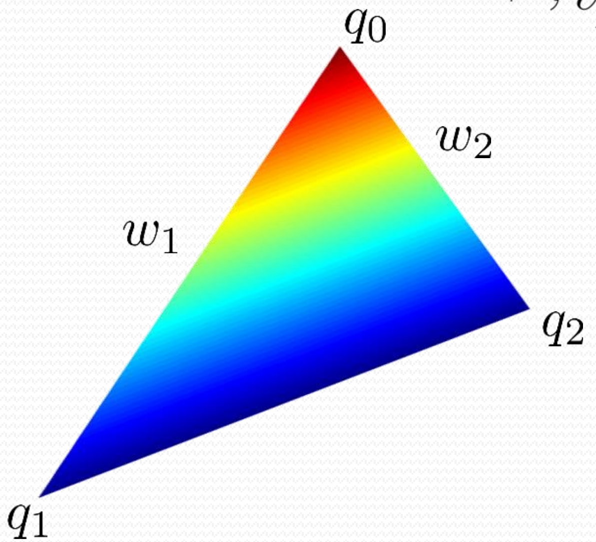
$$\begin{aligned} \nabla f(-1,0,0) &= d\mathbf{y} \cdot g_{\mathbf{y}}^{-1} \cdot \nabla \hat{f}(-1,0) \\ &= d\mathbf{y} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \end{aligned}$$

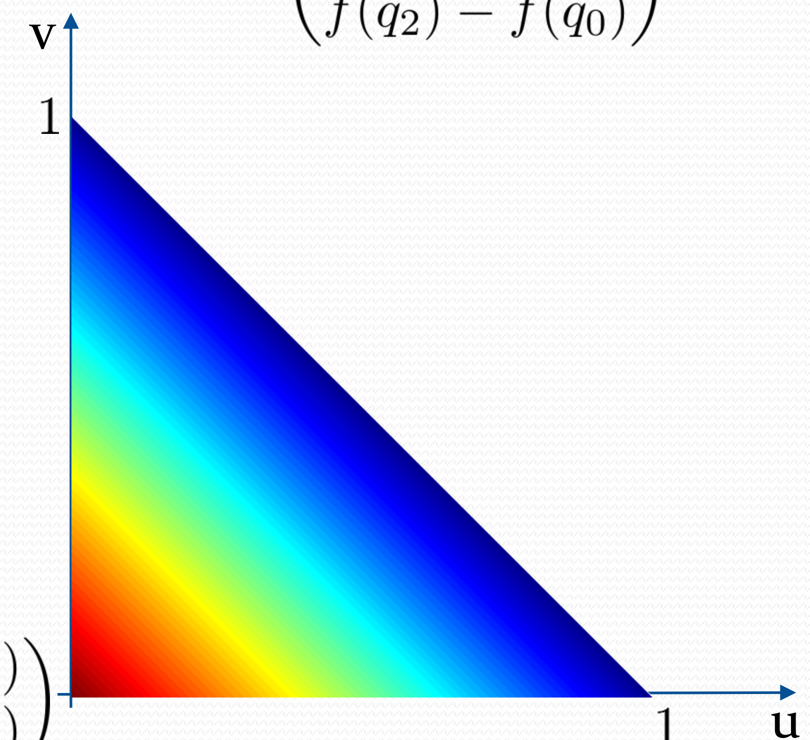
Gradient on a triangle

$$\nabla \tilde{f} \approx \begin{pmatrix} \tilde{f}(1,0) - \tilde{f}(0,0) \\ \tilde{f}(0,1) - \tilde{f}(0,0) \end{pmatrix}$$

$$= \begin{pmatrix} f(q_1) - f(q_0) \\ f(q_2) - f(q_0) \end{pmatrix}$$

$$\boldsymbol{x}(u,v) = q_0 + u w_1 + v w_2$$





$$\nabla f = \begin{pmatrix} \langle w_1, w_1 \rangle & \langle w_1, w_2 \rangle \\ \langle w_1, w_2 \rangle & \langle w_2, w_2 \rangle \end{pmatrix}^{-1} \nabla \tilde{f}$$

$$\approx \begin{pmatrix} \langle w_1, w_1 \rangle & \langle w_1, w_2 \rangle \\ \langle w_1, w_2 \rangle & \langle w_2, w_2 \rangle \end{pmatrix}^{-1} \begin{pmatrix} f(q_1) - f(q_0) \\ f(q_2) - f(q_0) \end{pmatrix}$$

Application of the gradient

Segmentation based on texture:

Main idea: Consider the norm of the gradient $\|\nabla f\|^2 = (\nabla \tilde{f})^T g^{-1} \nabla \tilde{f}$

Suggested reading

- *Differential geometry of curves and surfaces*. Do Carmo – Chapters 2.5, Appendix 2.B
- *Differential Geometry: Curves – Surfaces – Manifolds*. W. Kühnel – Chapter 3A