Analysis of Three-Dimensional Shapes
(IN2238, TU München, Summer 2014)

Isometries
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Seminar

«The metric approach to shape matching»

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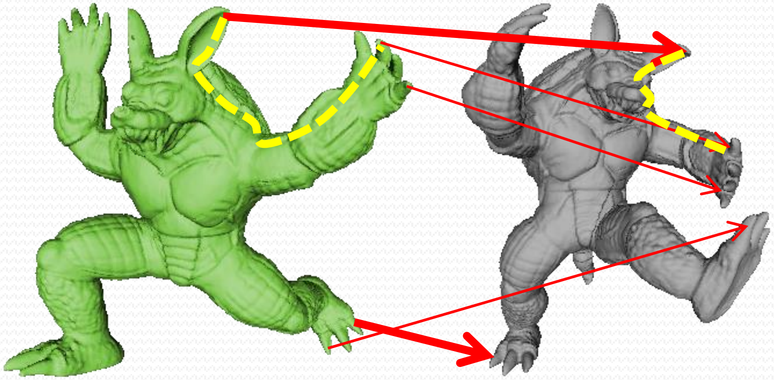
Wednesday, May 28th

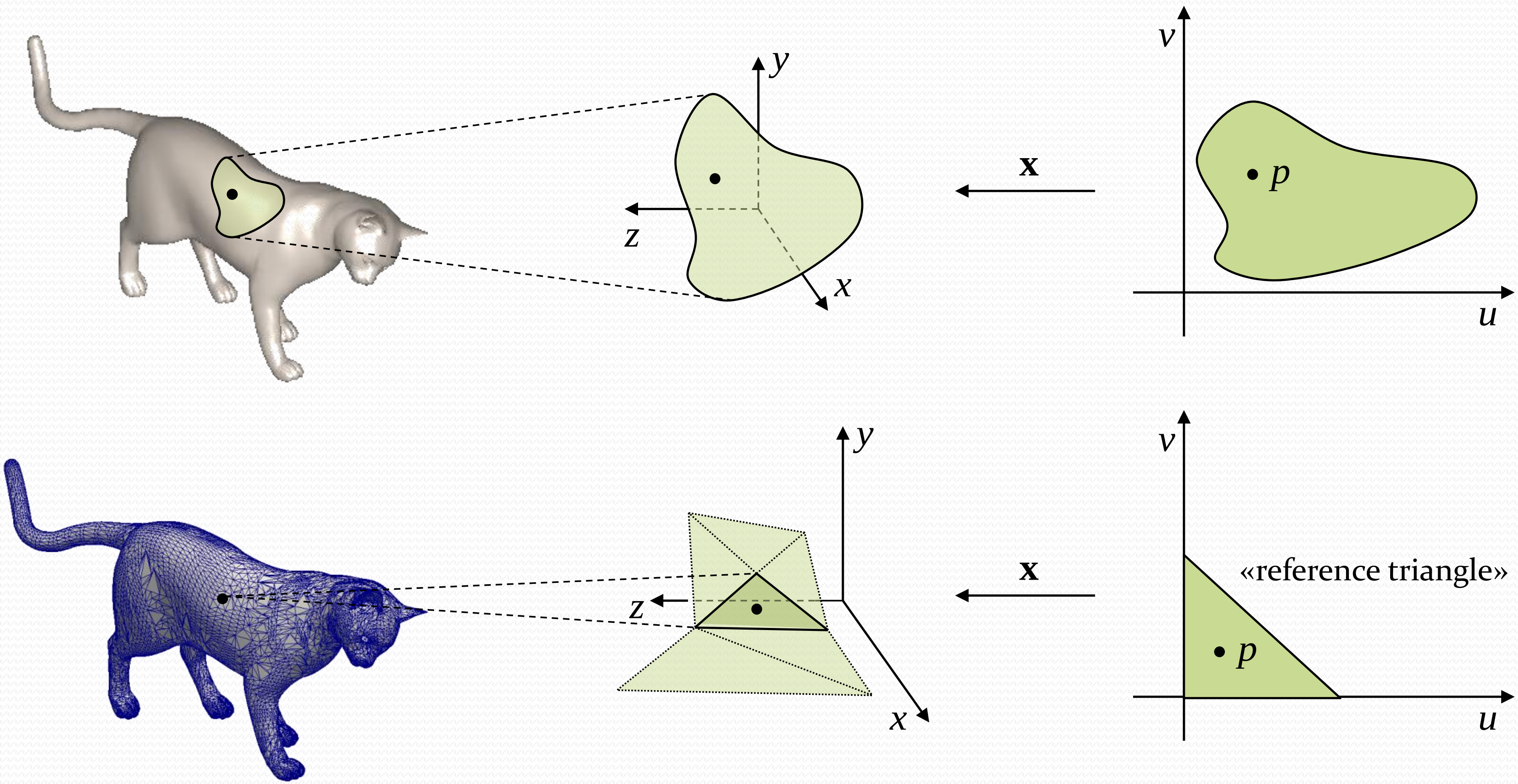
14:00 Room 02.09.023



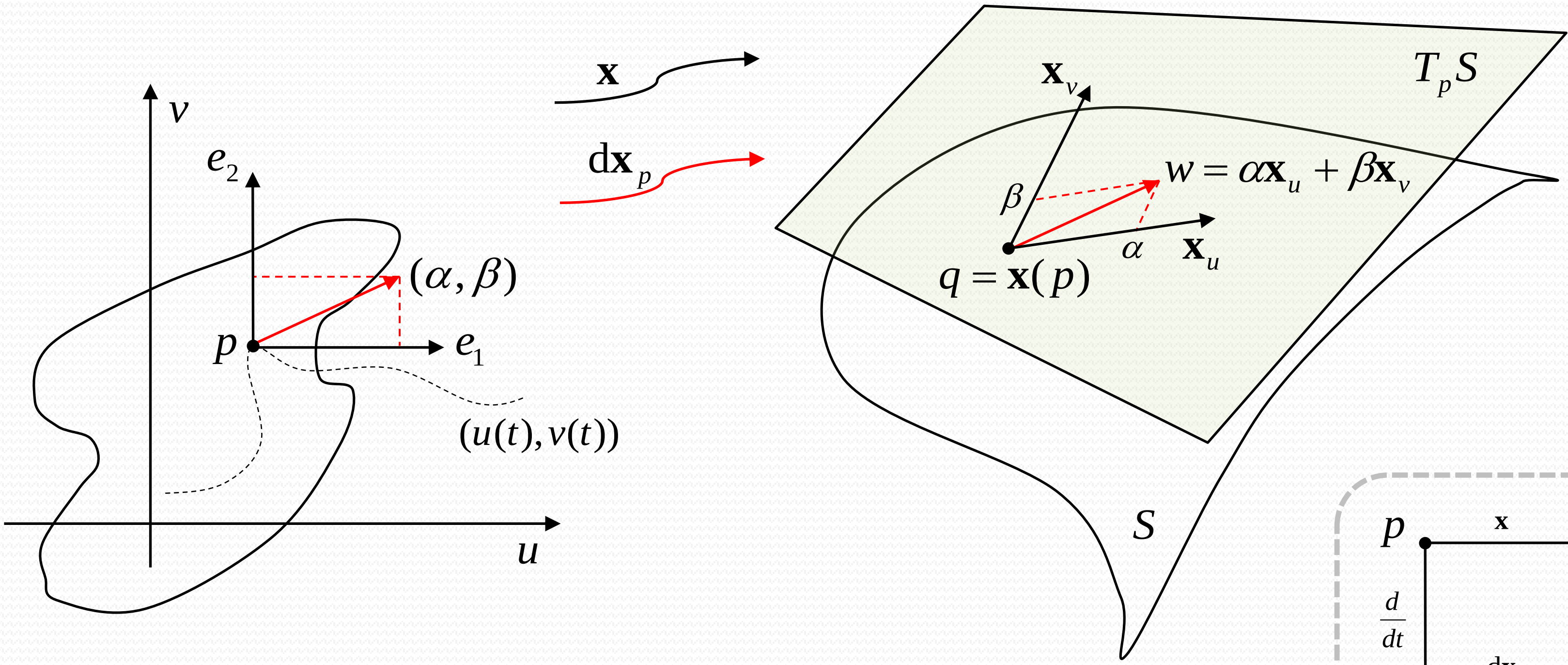
Overview

- Parametrized surfaces and first fundamental form
- Functions defined on surfaces
- Laplace-Beltrami operator
- Extension to triangulated manifolds

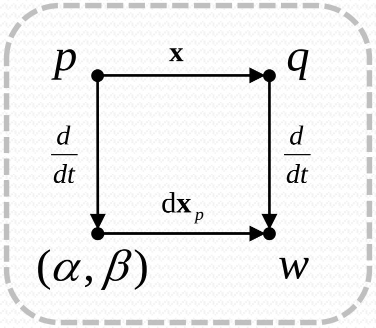
Wrap-up



Wrap-up



v
e_2
(α, β)
p
e_1
$(u(t), v(t))$
u
$\mathbf{x}$
$d\mathbf{x}_p$
$\mathbf{x}_v$
T_pS
$w = \alpha\mathbf{x}_u + \beta\mathbf{x}_v$
β
α
$\mathbf{x}_u$
$q = \mathbf{x}(p)$
S



p
$\mathbf{x}$
q
$\frac{d}{dt}$
$\frac{d}{dt}$
$d\mathbf{x}_p$
(α, β)
w

Wrap-up

We introduced a more generalized notion of first fundamental form on a regular surface, as the *bilinear* function $I_p : T_p(S) \times T_p(S) \to \mathbf{R}$ given by

$$I_p((\alpha,\beta),(\gamma,\delta)) = \langle \alpha \mathbf{x}_u + \beta \mathbf{x}_v, \gamma \mathbf{x}_u + \delta \mathbf{x}_v \rangle = \begin{pmatrix} \alpha & \beta \end{pmatrix} \underbrace{\begin{pmatrix} E & F \\ F & G \end{pmatrix}} \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$$

$$\mathrm{d}\mathbf{x}_p = \begin{pmatrix} \vdots & \vdots \\ \mathbf{x}_u & \mathbf{x}_v \\ \vdots & \vdots \end{pmatrix} \implies g = \mathrm{d}\mathbf{x}_p^{\mathrm{T}} \mathrm{d}\mathbf{x}_p$$

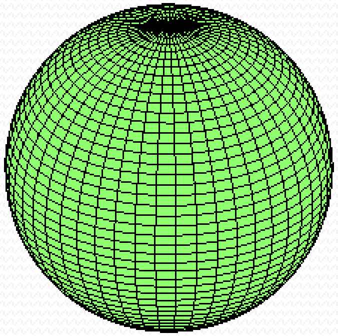
«metric tensor»

The metric tensor shows up when calculating quantities of interest on the surface, such as **areas**, **lengths**, **angles**, **gradients** of functions.

Wrap-up

The coefficients of the metric tensor are, in fact, *functions* defined on the surface element. Recall from this example:

$$\mathbf{x} : (0, 2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}^3 \qquad \mathbf{x}(u, v) = \begin{pmatrix} \cos(u)\cos(v) \\ \sin(u)\cos(v) \\ \sin(v) \end{pmatrix}$$



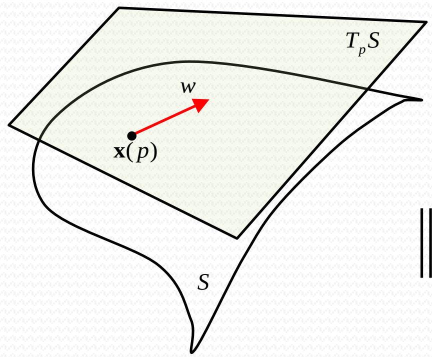
$$g = \mathrm{d}\mathbf{x}^{\mathrm{T}} \mathrm{d}\mathbf{x} = \begin{pmatrix} \cos^2(v) & 0 \\ 0 & 1 \end{pmatrix}$$

Thus, if $w = \alpha \mathbf{x}_u + \beta \mathbf{x}_v$ is the tangent vector to the sphere at point $\mathbf{x}(u,v)$, then its squared length is given by $|w|^2 = I(w) = \alpha^2 \cos^2(v) + \beta^2$.

Wrap-up

Length of a tangent vector

$w \equiv (\alpha, \beta)$

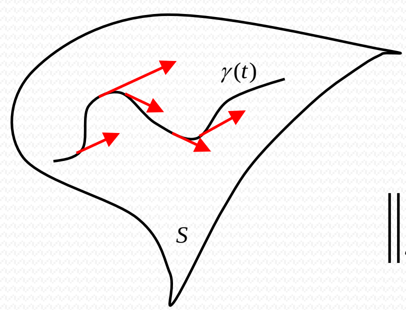


$$\|w\|^2 = \begin{pmatrix}\alpha & \beta\end{pmatrix} g \begin{pmatrix}\alpha \\ \beta\end{pmatrix}$$

Length of a curve

$\gamma(t) = \mathbf{x}(u(t), v(t))$

$\gamma : (0, T) \to S$

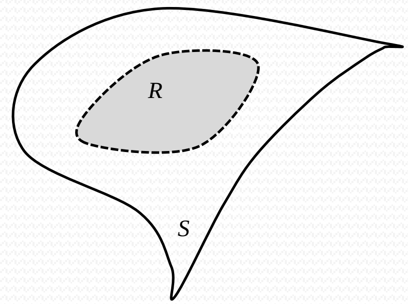


$$\|\gamma'(t)\| = \int_0^T \sqrt{\begin{pmatrix}u' & v'\end{pmatrix} g \begin{pmatrix}u' \\ v'\end{pmatrix}}\, dt$$

Wrap-up

Area of a region

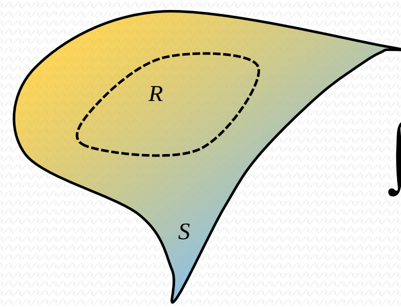
$Q = \mathbf{x}^{-1}(R)$



$$A(R) = \iint_Q \sqrt{\det g}\, du dv$$

Integral of a function

$f : S \to \mathbf{R}$

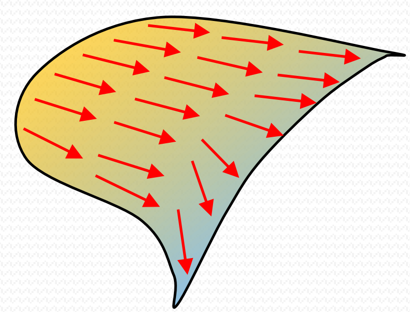


$$\int_R f(s) ds = \iint_Q f(\mathbf{x}(u,v)) \sqrt{\det g}\, du dv$$

Gradient of a function

$f : S \to \mathbf{R}$

$\tilde{f} = f \circ \mathbf{x}$



$$\nabla f = g^{-1} \nabla \tilde{f}$$

Wrap-up

We defined the **gradient** of a differentiable function $f : S \to \mathbb{R}$ as the *unique* vector field $\nabla f : S \to T_p S$ such that the following holds:

$$\langle \nabla f, \vec{v} \rangle = df_p(\vec{v})$$

directional derivative of f at p, along direction v

By giving this definition, we are implicitly applying the *Riesz representation theorem*, stated below.

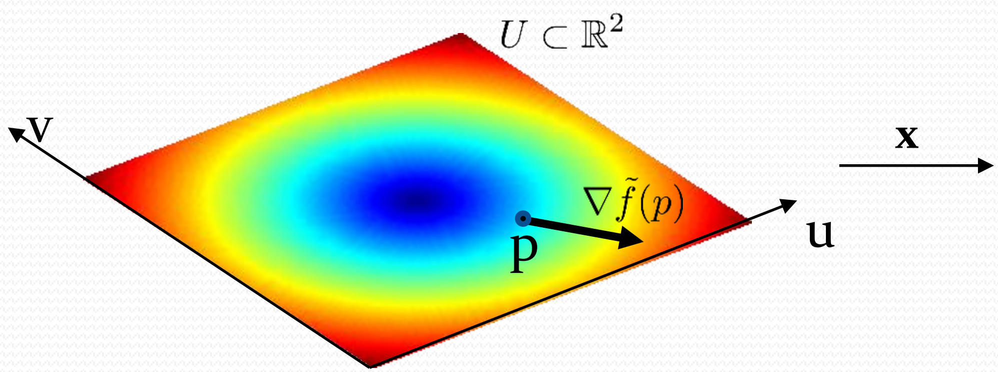
Let H be a Hilbert space, and let H^* be the space consisting of all continuous linear functions $\phi : H \to \mathbb{R}$ (the space H^* is also called the *dual space* of H). Then, *every* element of H^* can be written *uniquely* as an inner product:

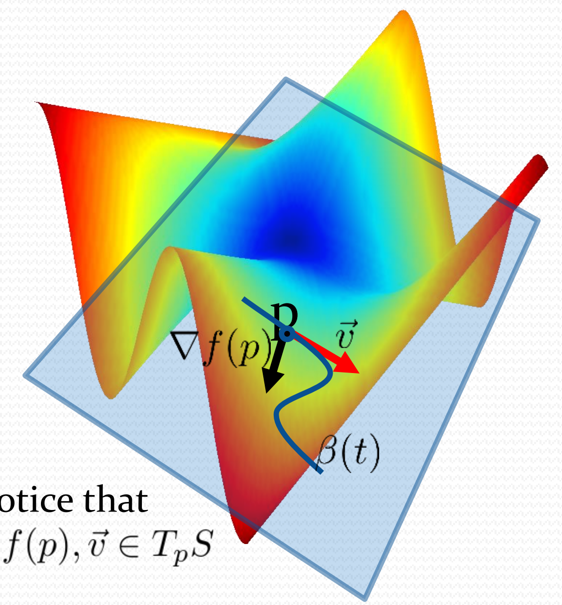
$$\phi(y) = \langle y, x \rangle \quad \forall y \in H$$

In our case, $H \equiv T_p S$ is the space of tangent vectors and $\phi \equiv df_p$ is the directional derivative of f at p (*do not confuse it with the differential* $\mathrm{d}\mathbf{x}_p$*!*)

Wrap-up

The definition we gave for the gradient is quite appropriate for our purposes, because we can transfer its computation to the parametrization domain U.





$$\nabla \tilde{f}(p) = \begin{pmatrix} \frac{\partial \tilde{f}}{\partial u}(p) \\ \frac{\partial \tilde{f}}{\partial v}(p) \end{pmatrix}$$ easy to compute!

notice that
$\nabla f(p), \vec{v} \in T_pS$

As with the previous quantities, the gradient will be independent of the specific parametrization **x** we choose.

Wrap-up

Thus, according to our definition of the gradient, we have to find the unique ∇f such that:

$$\langle \nabla f, \vec{v} \rangle = df_p(\vec{v})$$

Interestingly, by passing to local coordinates, we found that we can compute the directional derivative directly in U, as:

$$df_p(\vec{v}) = (\nabla \tilde{f})^T \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad \text{where} \quad \tilde{f} = f \circ \mathbf{x}$$

defined on
the surface S

defined on
the parameter
domain U

We can thus write $\langle \nabla f, \vec{v} \rangle = (\nabla \tilde{f})^T \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

Wrap-up

Using the «bilinear» definition of first fundamental form, we can also write

$$\langle \nabla f, \vec{v} \rangle = I_p(\nabla f, \vec{v}) = \begin{pmatrix} f_1 & f_2 \end{pmatrix} g \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Together with the last equation from the previous slide, we have

$$(\nabla \tilde{f})^T \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} f_1 & f_2 \end{pmatrix} g \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

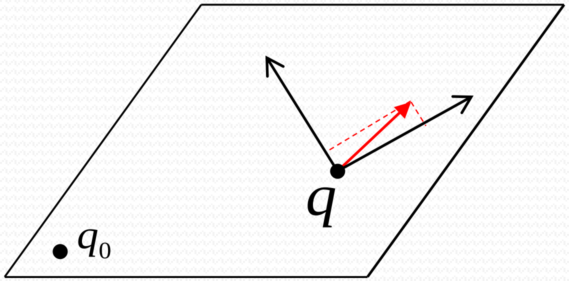
And thus we can finally obtain:

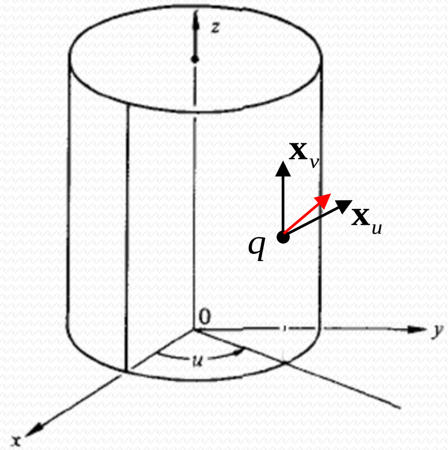
$$\nabla f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = g^{-1} \nabla \tilde{f}$$

Isometries

We have already seen that plane and cylinder behave locally in the same way, since their metric tensors are equal (at least on the surface elements we considered).

We captured this behavior by saying that plane and cylinder are «locally isometric». We will now give a more formal definition for **isometry**, and we will link it to the notion we already have from metric geometry.





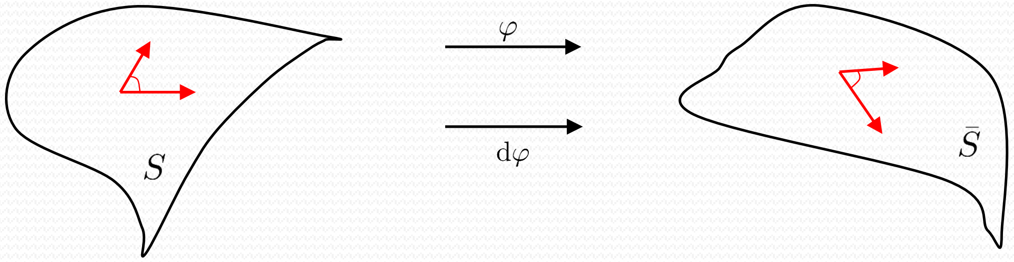
Isometries

A diffeomorphism $\varphi : S \to \bar{S}$ is called an **isometry** if

$$\langle w_1, w_2 \rangle = \langle \mathrm{d}\varphi_p(w_1), \mathrm{d}\varphi_p(w_2) \rangle$$

for all $p \in S$ and all pairs of tangent vectors $w_1, w_2 \in T_pS$.

In other words, a diffeomorphism φ is an isometry if its associated differential $\mathrm{d}\varphi$ preserves the inner product.



Isometries

A direct consequence is that <u>isometries preserve the first fundamental form</u>:

$$I_p(w) \underset{\substack{\text{first}\\ \text{fundamental}\\ \text{form on } S}}{=} \langle w, w \rangle \underset{\text{isometry}}{=} \langle \mathrm{d}\varphi_p(w), \mathrm{d}\varphi_p(w) \rangle \underset{\substack{\text{first}\\ \text{fundamental}\\ \text{form on } \bar{S}}}{=} I_{\varphi(p)}(\mathrm{d}\varphi_p(w))$$

for all $w \in T_pS$.

The converse is also true. If a diffeomorphism φ preserves the first fundamental form, then it is an isometry:

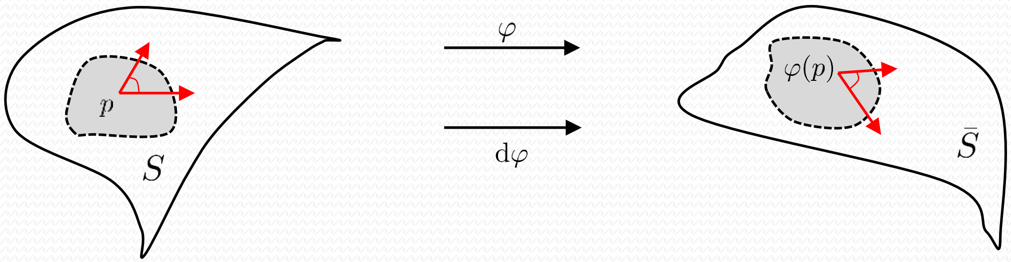
$$\begin{aligned}
2\langle w_1, w_2 \rangle &= I_p(w_1 + w_2) - I_p(w_1) - I_p(w_2) \\
&= I_{\varphi(p)}(\mathrm{d}\varphi_p(w_1 + w_2)) - I_{\varphi(p)}(\mathrm{d}\varphi_p(w_1)) - I_{\varphi(p)}(\mathrm{d}\varphi_p(w_2)) \\
&= 2\langle \mathrm{d}\varphi_p(w_1), \mathrm{d}\varphi_p(w_2) \rangle
\end{aligned}$$

by assumption that φ preserves the first fundam. form

Local isometries

Note that the definition we gave for isometry requires φ to be a diffeomorphism.

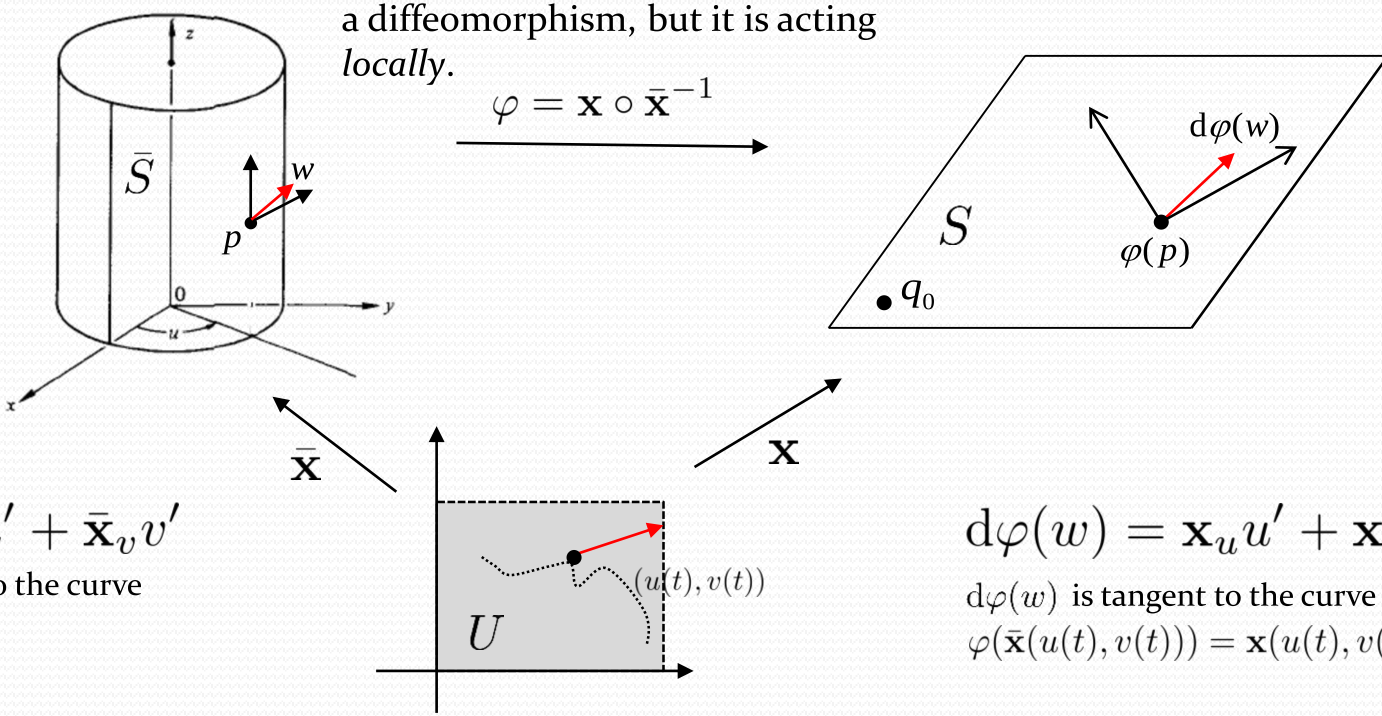
If this is not the case, then a map $\varphi : V \to \bar{S}$ of a neighborhood V of $p \in S$ is called a **local isometry** at p if there exists a neighborhood $\bar{V}$ of $\varphi(p) \in \bar{S}$ such that $\varphi(p) : V \to \bar{V}$ is an isometry.



If there exists a local isometry at every $p \in S$, the surface S is said to be **locally isometric** to $\bar{S}$.

Example

Note that $\varphi : \bar{\mathbf{x}}(U) \to \mathbf{x}(U)$ is indeed a diffeomorphism, but it is acting *locally*.



$$w = \bar{\mathbf{x}}_u u' + \bar{\mathbf{x}}_v v'$$

w is tangent to the curve $\bar{\mathbf{x}}(u(t), v(t))$

$$\mathrm{d}\varphi(w) = \mathbf{x}_u u' + \mathbf{x}_v v'$$

$\mathrm{d}\varphi(w)$ is tangent to the curve $\varphi(\bar{\mathbf{x}}(u(t), v(t))) = \mathbf{x}(u(t), v(t))$

As we have seen previously, we can get the same metric tensor for $\bar{\mathbf{x}}$ and $\mathbf{x}$. Thus, the two surfaces are *locally isometric*, since it holds:

$$I_p(w) = \bar{E}(u')^2 + 2\bar{F}u'v' + \bar{G}(v')^2 = E(u')^2 + 2Fu'v' + G(v')^2 = I_{\varphi(p)}(\mathrm{d}\varphi_p(w))$$

Local isometries

The previous example can be generalized to get an important result:

If two parametrizations **x** and $\bar{\mathbf{x}}$ give rise to the same metric tensor, then the composition $\varphi = \mathbf{x} \circ \bar{\mathbf{x}}^{-1}$ is a local isometry. The proof follows the same steps as in the previous example.

The converse is also true. That is, if a map $\varphi : S \to \bar{S}$ is an <u>isometry</u> and $\mathbf{x} : U \to S$ is a parametrization at $p \in S$, then $\bar{\mathbf{x}} = \varphi \circ \mathbf{x}$ is a parametrization at $\varphi(p)$ and the metric tensors g and $\bar{g}$ are equal.

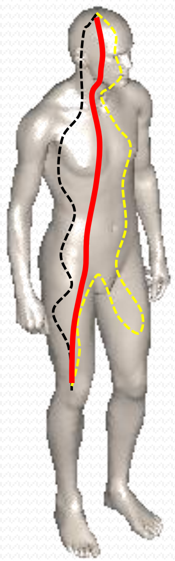
Intrinsic distance

We have seen how to use the first fundamental form to measure lengths of paths on a surface. This allows us to introduce a notion of «intrinsic» distance for points on the surface.

We define the distance $d(p,q)$ between two points of S as

$$d(p,q) = \inf_{\alpha:[0,1]\to S} \int_0^1 \|\alpha'(t)\| dt$$

where $\alpha(0) = p,\ \alpha(1) = q\,.$



According to this definition, every regular surface comes with a «natural» metric induced by the first fundamental form (the fact that d defined above is actually a metric should be proven, but we will not do it here).

Isometries: equivalence of the definitions

The distance *d* is invariant under isometries, that is, if $\varphi : S \to \bar{S}$ is an isometry, then

$$d(p,q) = d(\varphi(p), \varphi(q))$$

for all $p, q \in S$.

From this proposition it seems like our original notion of isometry (i.e. from the point of view of metric spaces) is just a consequence of the new «differential» definition we gave in the previous slides.

In fact, we will now show that *the two definitions are equivalent* if we consider the natural, intrinsic metric induced by the first fundamental form.

Equivalence (1/2)

$$d(p,q) = \inf_{\alpha:[0,1]\to S} \int_0^1 \|\alpha'(t)\| dt = \inf_{\alpha:[0,1]\to S} \int_0^1 \sqrt{I_p(w(t))} dt$$

If $\varphi : S \to \bar{S}$ is an isometry, then $I_p(w) = I_{\varphi(p)}(\mathrm{d}\varphi_p(w))$ and thus by integrating we get:

$$\int_0^1 \sqrt{I_p(w(t))} dt = \int_0^1 \sqrt{I_{\varphi(p)}(\mathrm{d}\varphi_p(w(t)))} dt$$

In particular, the infimum will also have the same value. As a consequence,

$$\inf_{\alpha:[0,1]\to S} \int_0^1 \|\alpha'(t)\| dt = \inf_{\varphi(\alpha):[0,1]\to \bar{S}} \int_0^1 \|\varphi'(\alpha(t))\| dt$$

$\boxed{\beta : [0,1] \to \bar{S}}$ identifies the same set

$\Downarrow$

$$d(p,q) = d(\varphi(p), \varphi(q))$$

Equivalence (2/2)

Let us now assume that $\varphi : S \to \bar{S}$ is such that, for all pairs of points *p*,*q*:

$$d(p, q) = d(\varphi(p), \varphi(q))$$

This means that there exist two curves of equal length attaining the infima (this follows from Hopf-Rinow theorem, which we won't cover):

$$\inf_{\alpha:[0,1]\to S} \int_0^1 \|\alpha'(t)\| dt = \inf_{\beta:[0,1]\to \bar{S}} \int_0^1 \|\beta'(t)\| dt$$

where $\alpha(0) = p,\ \alpha(1) = q$ and $\beta(0) = \varphi(p),\ \beta(1) = \varphi(q)$.

It is not difficult to show that one can find two parametrizations $\mathbf{x}$, $\bar{\mathbf{x}}$ such that

$$\varphi = \bar{\mathbf{x}} \circ \mathbf{x}^{-1}$$

But then, following the reasoning from the cylinder/plane example, this means that $\beta = \varphi(\alpha)$ and that the first fundamental forms on *S* and $\bar{S}$ are the same.

Hence, φ is an isometry.

Suggested reading

- *Differential geometry of curves and surfaces*. Do Carmo – Chapters 4.1, 4.2, Exercises 2, 3, 9, 18