Analysis of Three-Dimensional Shapes

(IN2238, TU München, Summer 2014)

The Laplacian
(26.05.2014)

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Seminar

«The metric approach to shape matching»

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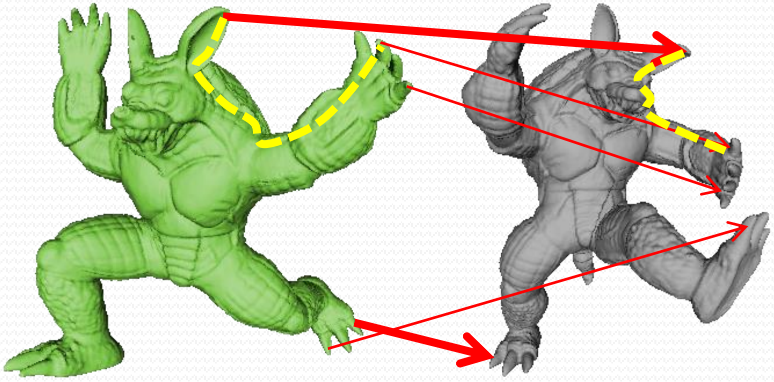
Wednesday, May 28th

14:00 Room 02.09.023

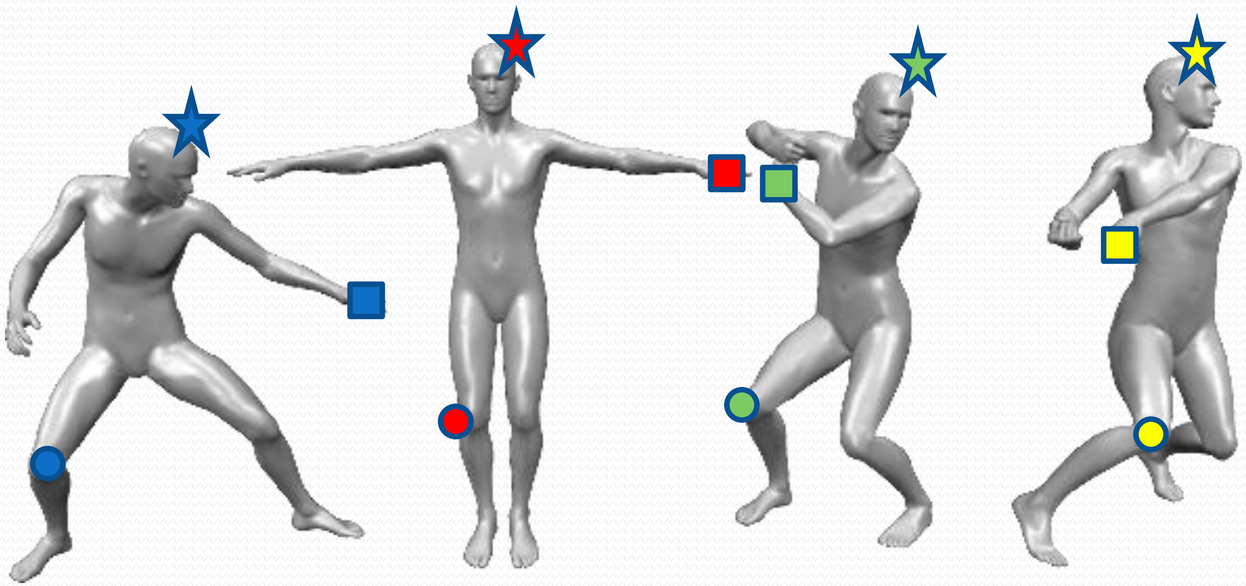


Overview

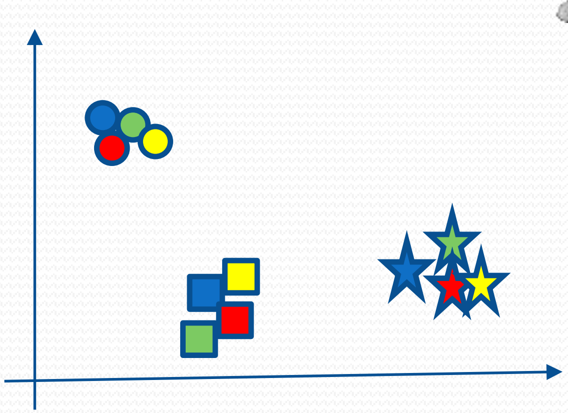
- Parametrized surfaces and first fundamental form
- Functions defined on surfaces
- Laplace-Beltrami operator
- Extension to triangulated manifolds

Point descriptors

Isometric shapes



Point descriptor



Consider physical phenomena

Heat diffusion in R^n

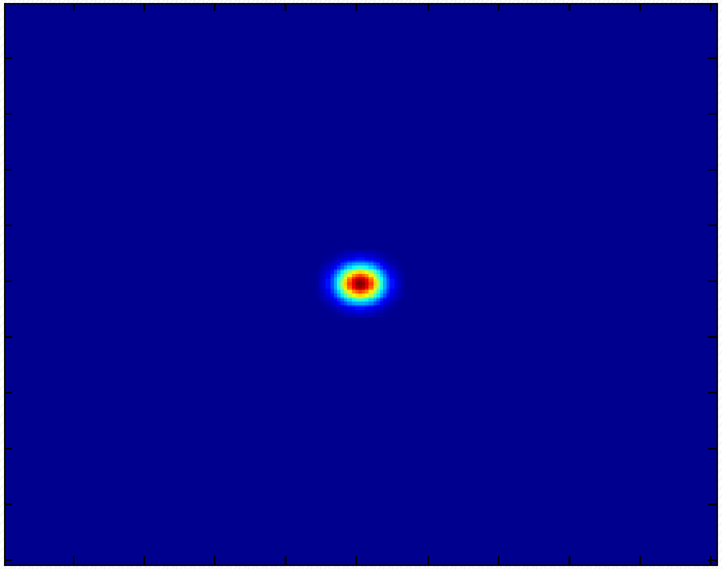
For an open subset $U \subset \mathbb{R}^n$ the diffusion of heat is described by the heat equation:

$$\frac{\partial \tilde{u}(x,t)}{\partial t} = \Delta \tilde{u}(x,t)$$

$$\tilde{u}(x,0) = \tilde{u}_0(x)$$

$$\tilde{u}(\partial U, t) = \ldots$$

$$\Delta \tilde{u}(x,t) = \operatorname{div}(\nabla \tilde{u}) = \sum_{i=1}^{n} \frac{\partial^2 \tilde{u}(x,t)}{\partial x_i^2}$$



Heat diffusion on surfaces

For a regular surface S the diffusion of heat is described by the heat equation:

$$\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t)$$

$$u(x,0) = u_0(x)$$

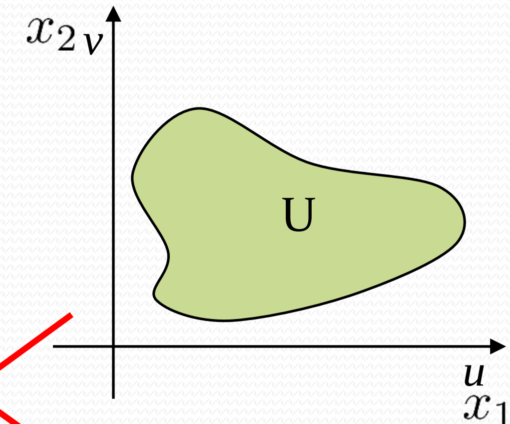
$$u(\partial U, t) = \ldots$$

$$\Delta u(x,t) = \operatorname{div}(\nabla u) = ?$$

The divergence in R^n

Like last week: indirect definition of the divergence:

Let $\tilde{f} : U \to \mathbb{R}, \tilde{\vec{V}} = (\tilde{V}_1, \tilde{V}_2, ..., \tilde{V}_n) : U \to \mathbb{R}^n$ be sufficiently smooth and $\tilde{f}$ vanishing on the boundary of $U \subset \mathbb{R}^n$



$$\langle \nabla \tilde{f}, \tilde{\vec{V}} \rangle_U = \int_U \langle \nabla \tilde{f}(x), \tilde{\vec{V}}(x) \rangle dx$$

$$= \int_U \sum_i \frac{\partial \tilde{f}(x)}{\partial x_i} \cdot \tilde{V}_i(x) dx$$

(partial integration)

$$= -\int_U \sum_i \tilde{f}(x) \cdot \frac{\partial \tilde{V}_i(x)}{\partial x_i} dx + \cancel{\int_{\partial U} \cdots}$$

$$= -\int_U \tilde{f}(x) \underbrace{\sum_i \frac{\partial \tilde{V}_i(x)}{\partial x_i}}_{\mathrm{div}(\tilde{\vec{V}})} dx = -\langle \tilde{f}, \mathrm{div} \tilde{\vec{V}} \rangle_U$$

The divergence on a surface

Let S be a regular surface without boundary and $V(p) = V_1 \mathbf{x}_u + V_2 \mathbf{x}_v \in T_pS$. We define $\text{div } V : S \to \mathbb{R}$ via

$$\int_S f(p)\text{div } V(p) dS = -\int_S I_p(\nabla f(p), V(p)) dS \quad \forall f \in C^\infty(S)$$

Exercise: The divergence is a linear operator.

Let us next write the divergence in local coordinates. The left term becomes:

$$\int_S f(p)\text{div } V(p) dS = \int_U \tilde{f}(x)\text{div } V(x)\sqrt{\det g(x)}dx$$

This is the quantity we are interested in!

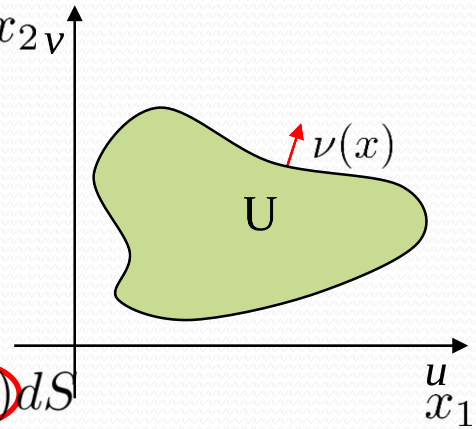
Integration by parts

Let $\tilde{f}$ be smooth but **not vanishing** on the boundary of $U \subset \mathbb{R}^n$

$$\langle \nabla \tilde{f}, \vec{\tilde{V}} \rangle_U = \int_U \langle \nabla \tilde{f}(x), \vec{\tilde{V}}(x) \rangle dx$$

$$= \int_U \sum_i \frac{\partial \tilde{f}(x)}{\partial x_i} \cdot \tilde{V}_i(x) dx$$

$$= -\int_U \sum_i \tilde{f}(x) \cdot \frac{\partial \tilde{V}_i(x)}{\partial x_i} dx + \int_{\partial U} \tilde{f}(x) \vec{\tilde{V}}(x) \cdot \vec{\nu}(x) dS$$



$$= -\langle \tilde{f}, \mathrm{div} \vec{\tilde{V}} \rangle_U + \int_{\partial U} \tilde{f}(x) \vec{\tilde{V}}(x) \cdot \vec{\nu}(x) dS$$

Divergence in local coordinates

The right term reads:

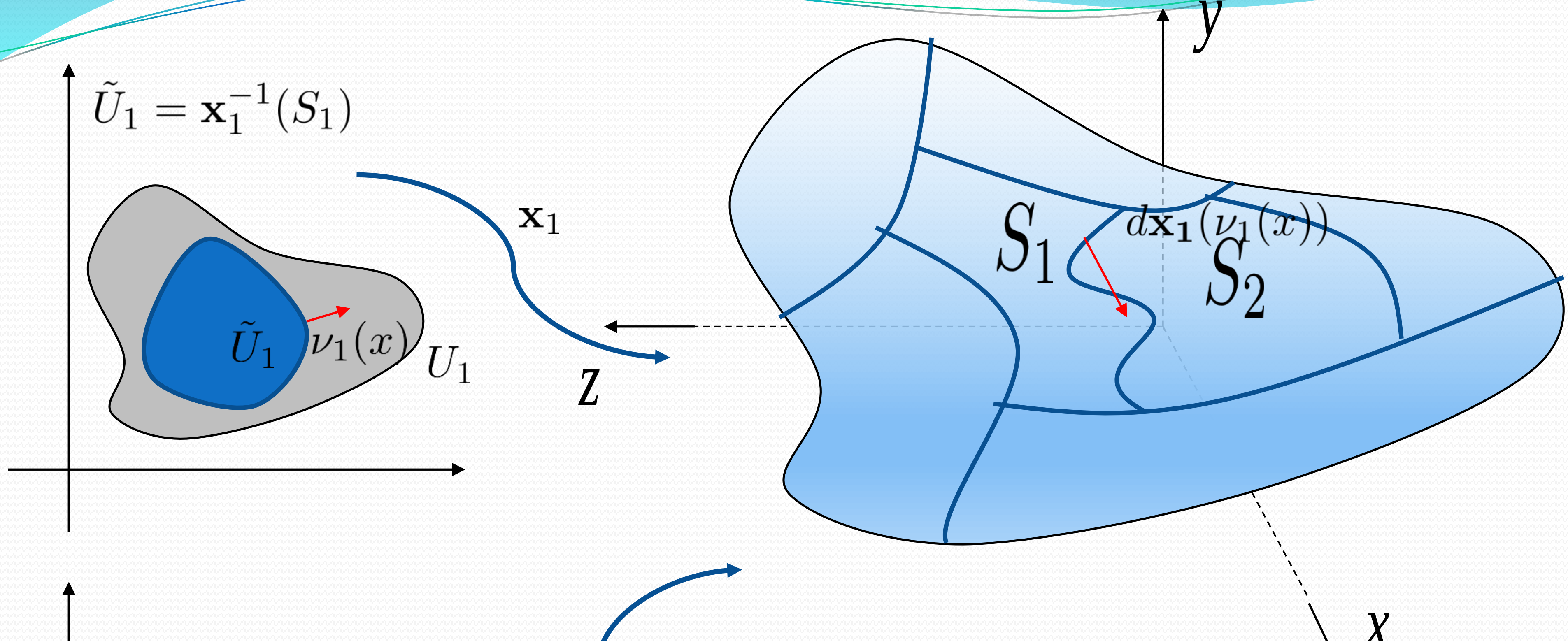
$$-\int_S I_p(\nabla f, V)\,dS = -\sum_j \int_{\tilde{U}_j} (g_j^{-1}\nabla \tilde{f}_j)^T g_j \vec{\tilde{V}}_j \sqrt{\det g_j}\ dx$$

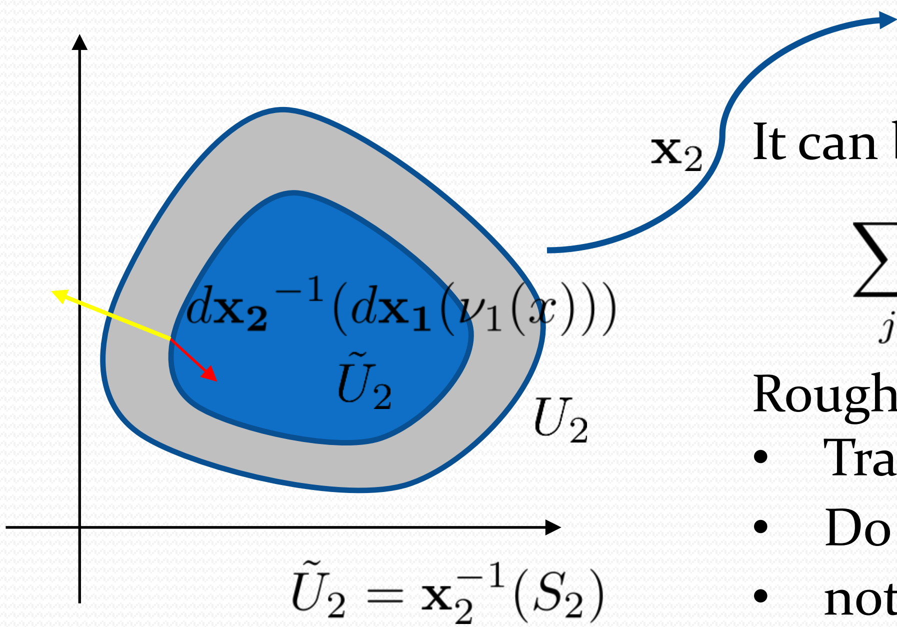
$$= -\sum_j \int_{\tilde{U}_j} (\nabla \tilde{f}_j)^T \vec{\tilde{V}}_j \sqrt{\det g_j}\,dx$$

$$= -\sum_j \langle \nabla \tilde{f}_j, \sqrt{\det g_j}\vec{\tilde{V}}_j \rangle_{\tilde{U}_j}$$

$$= \sum_j \langle \tilde{f}_j, \operatorname{div}(\sqrt{\det g_j}\vec{\tilde{V}}_j)\rangle_{\tilde{U}_j} - \sum_j \int_{\partial \tilde{U}_j} f_j(x)\vec{\tilde{V}}_j(x)\sqrt{\det g_j(x)}\nu_j(x)\,dx$$

No open subset of R² is diffeomorphic to a surface without boundary





It can be shown that

$$\sum_j \int_{\partial \tilde{U}_j} f_j(x) \vec{\tilde{V}}_j(x) \sqrt{\det g_j(x)}\, \nu_j(x) dx = 0$$

Rough idea:

- Track normal vector from domain to domain
- Do the same for vector field $\vec{V}$
- notice that $d\mathbf{x}_1(\vec{\tilde{V}}_1) = d\mathbf{x}_2(\vec{\tilde{V}}_2) = \vec{V}$

Divergence in local coordinates

$$-\int_S I_p(\nabla f, V)\,dS = -\sum_j \int_{\tilde{U}_j} (g_j^{-1}\nabla \tilde{f}_j)^T g_j \vec{\tilde{V}}_j \sqrt{\det g_j}\,dx$$

$$= -\sum_j \int_{\tilde{U}_j} (\nabla \tilde{f}_j)^T \vec{\tilde{V}}_j \sqrt{\det g_j}\,dx$$

$$= -\sum_j \langle \nabla \tilde{f}_j, \sqrt{\det g_j}\,\vec{\tilde{V}}_j \rangle_{\tilde{U}_j}$$

$$= \sum_j \langle \tilde{f}_j, \operatorname{div}(\sqrt{\det g_j}\,\vec{\tilde{V}}_j) \rangle_{\tilde{U}_j} - \xcancel{\sum_j \int_{\partial \tilde{U}_j} f_j(x) \vec{\tilde{V}}_j(x) \sqrt{\det g_j(x)}\,\nu_j(x)\,dx}$$

$$= \int_U \tilde{f}(x) \sum_i \frac{\partial}{\partial x_i}\left(\sqrt{\det g(x)}\,\tilde{V}_i(x)\right)dx = \int_U \tilde{f}(x)\operatorname{div} V(x)\sqrt{\det g(x)}\,dx$$

$$U = \bigcup \tilde{U}_j$$
$$\tilde{f}|_{\tilde{U}_j} = \tilde{f}_j$$
$$\tilde{V}|_{\tilde{U}_j} = \tilde{V}_j$$
$$\tilde{g}|_{\tilde{U}_j} = \tilde{g}_j$$

Since this equality has to hold for every *f*, we can deduce:

$$\operatorname{div} V = \frac{1}{\sqrt{\det g(x)}} \sum_i \frac{\partial}{\partial x_i}\left(V_i(x)\sqrt{\det g(x)}\right)$$

The Laplacian in local coordinates

Now that we know how to write the gradient and the divergence in local coordinates, we can combine those two to get an expression for the Laplace-Beltrami operator. We consider the special vector field $V = \nabla f$ and plug it in the formula for the divergence:

$$\begin{aligned}\Delta f = \text{div } V &= \frac{1}{\sqrt{\det g}} \sum_i \frac{\partial}{\partial x_i} \left((\nabla f(x))_i \sqrt{\det g(x)} \right) \\ &= \frac{1}{\sqrt{\det g}} \sum_i \frac{\partial}{\partial x_i} \left((g^{-1} \nabla \tilde{f}(x))_i \sqrt{\det g(x)} \right) \\ &= \frac{1}{\sqrt{\det g}} \sum_i \frac{\partial}{\partial x_i} \left(\sum_j g^{ij} \frac{\partial \tilde{f}(x)}{\partial x_j} \sqrt{\det g(x)} \right) \\ &= \frac{1}{\sqrt{\det g}} \sum_{i,j} \frac{\partial}{\partial x_i} \left(g^{ij} \frac{\partial \tilde{f}(x)}{\partial x_j} \sqrt{\det g(x)} \right)\end{aligned}$$

$$g^{-1} = \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix}$$

Exercise: The Laplacian is a linear operator.

Eigenvalues and eigenvectors

From linear algebra we know eigenvalues and eigenvectors of a matrix $A \in \mathbb{C}^{n \times n}$

A vector $0 \neq v \in \mathbb{C}^n$ is called eigenvector of A if there exists a $\lambda \in \mathbb{C}$ such that $Av = \lambda v$.

If v and w are eigenvectors to the eigenvalue λ, then $(\alpha v+\beta w)$ is an eigenvector to the same eigenvalue for all $\alpha, \beta \neq 0$.

A matrix is called Hermitian if $\langle Av, w\rangle = \langle v, Aw\rangle \quad \forall v, w \in \mathbb{C}^n$

Theorem
If *A* is Hermitian, then all its eigenvalues are real and eigenvectors to distinct eigenvalues are orthogonal.

Consequence
We can find an orthonormal basis consisting of eigenvectors.

The ONB is not unique!

Helmholtz equation

Theorem

The eigenvalues of the Laplacian

$$\Delta f = \lambda f$$

1. are real
2. Non positive
3. Eigenvectors to distinct eigenvalues are orthogonal

These are direct consequences of

$$\langle \Delta f, g \rangle = \langle \text{div } \nabla f, g \rangle = -\langle \nabla f, \nabla g \rangle = \langle f, \text{div } \nabla g \rangle = \langle f, \Delta g \rangle$$

We will just show the second property:

$$\Delta f = \lambda f \Rightarrow \lambda \langle f, f \rangle = \langle \lambda f, f \rangle = \langle \Delta f, f \rangle = -\langle \nabla f, \nabla f \rangle$$

As a side effect, we see that exactly one eigenvalue vanishes, with the constant function as the corresponding eigenfunction.

Discrete spectrum

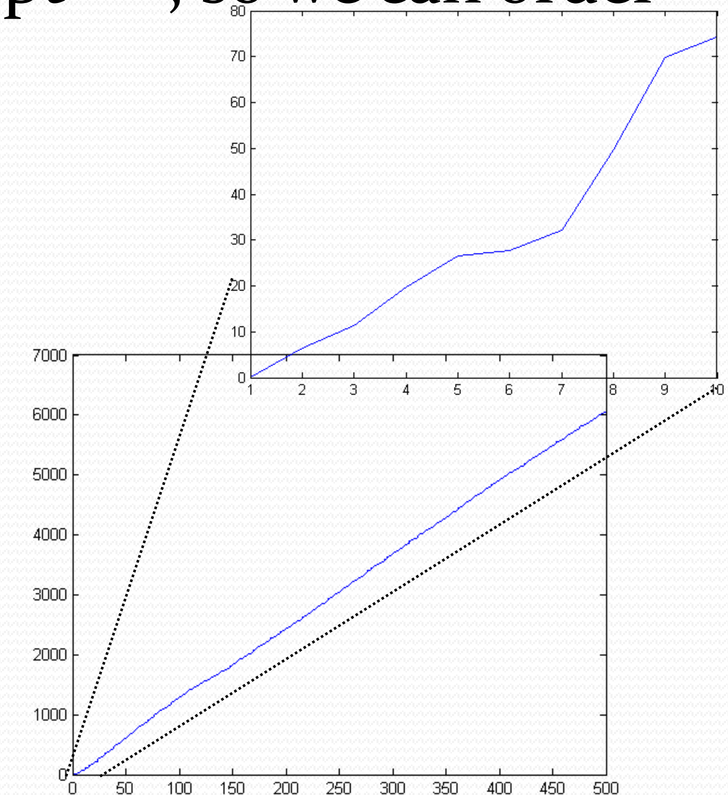
It can be shown, that the eigenvalues of the Laplacian defined on a compact surface without boundary are countable with no limit-point except -∞, so we can order them:

$$0 = \lambda_0 > \lambda_1 \geq \lambda_2 \geq \ldots \to -\infty$$

Weyl's law

$$-\lambda_j \sim \frac{\pi}{|S|} j \qquad \text{for } j \to \infty$$

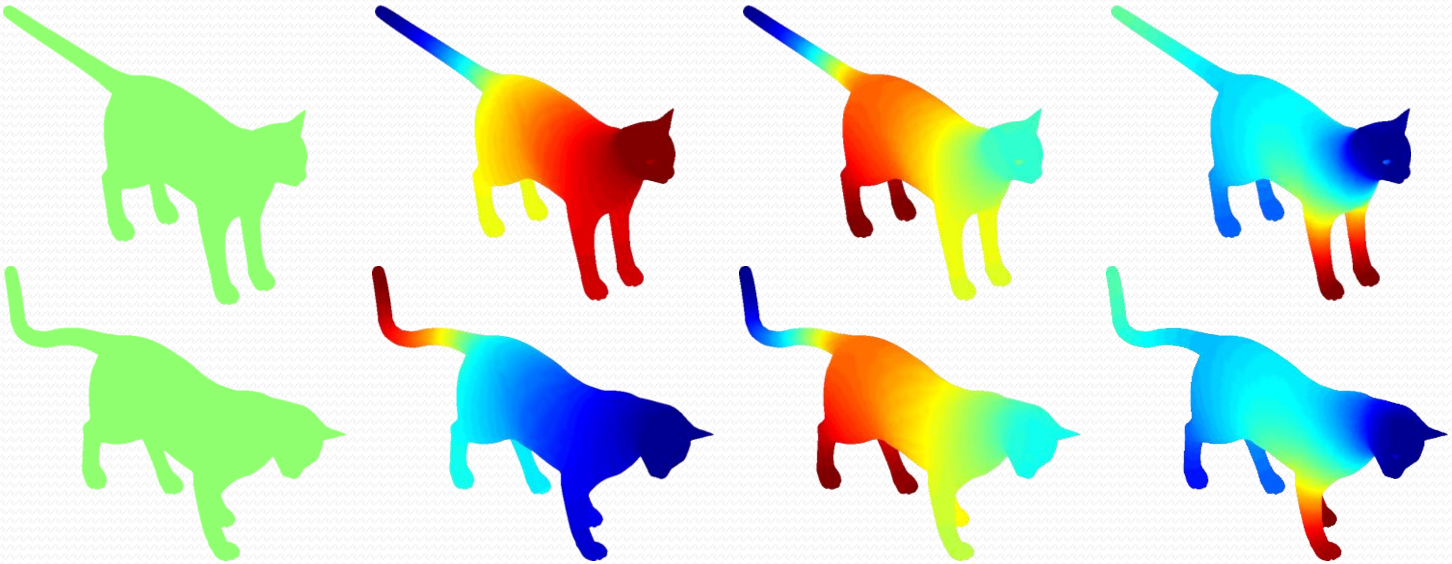
Note, that we can not say much about the multiplicity of eigenvalues.



Invariance under isometries

$$\Delta f = \frac{1}{\sqrt{\det g}} \sum_{i,j} \frac{\partial}{\partial x_i} \left(g^{ij} \frac{\partial \tilde{f}(x)}{\partial x_j} \sqrt{\det g(x)} \right)$$

The Laplace-Beltrami operator only depends on the metric tensor g. It is therefore invariant under isometric deformations of the surface.

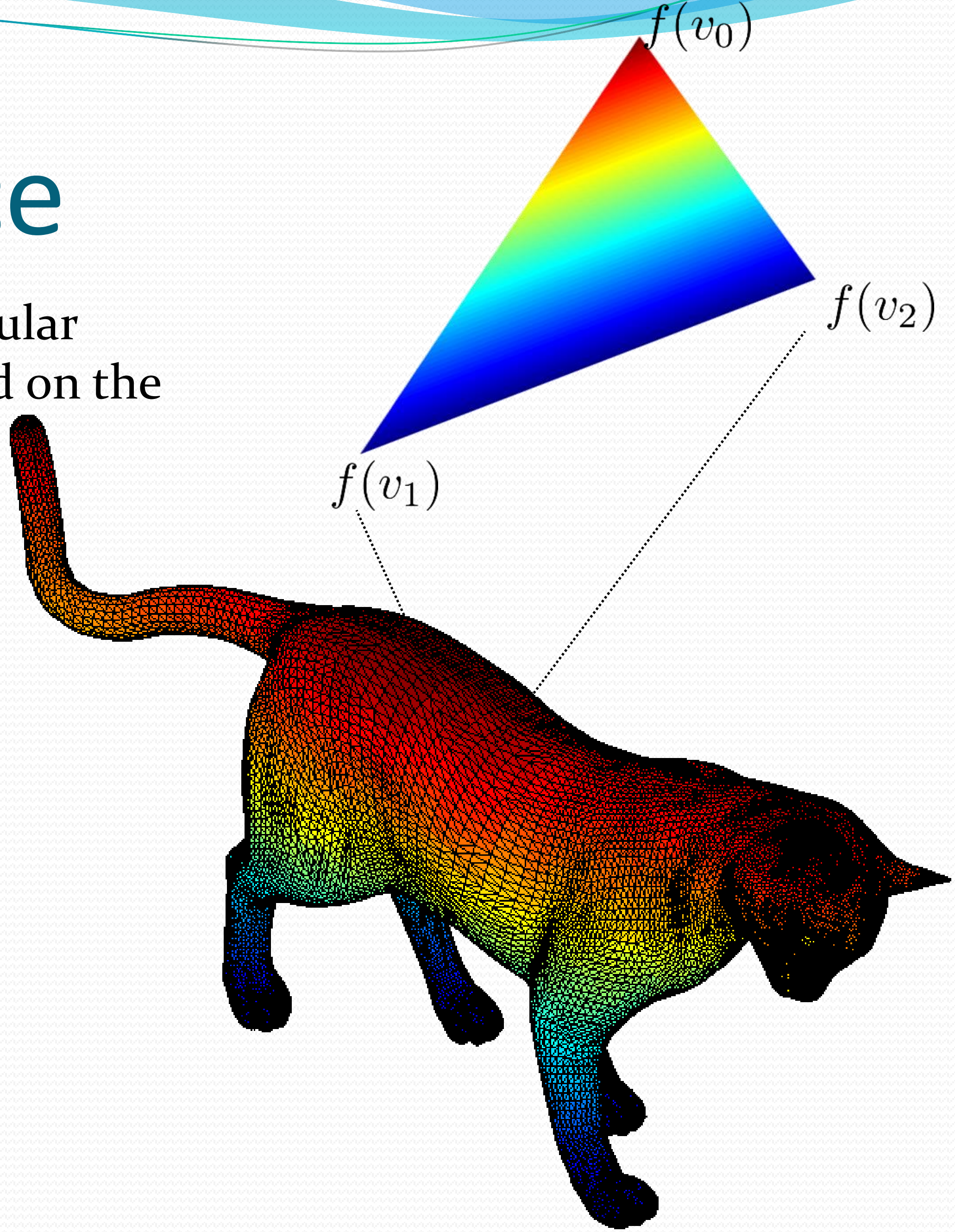


Discrete surface

Assume now we are given a triangular mesh with function values defined on the vertices.

Question:
How do we define function values inside a triangle?

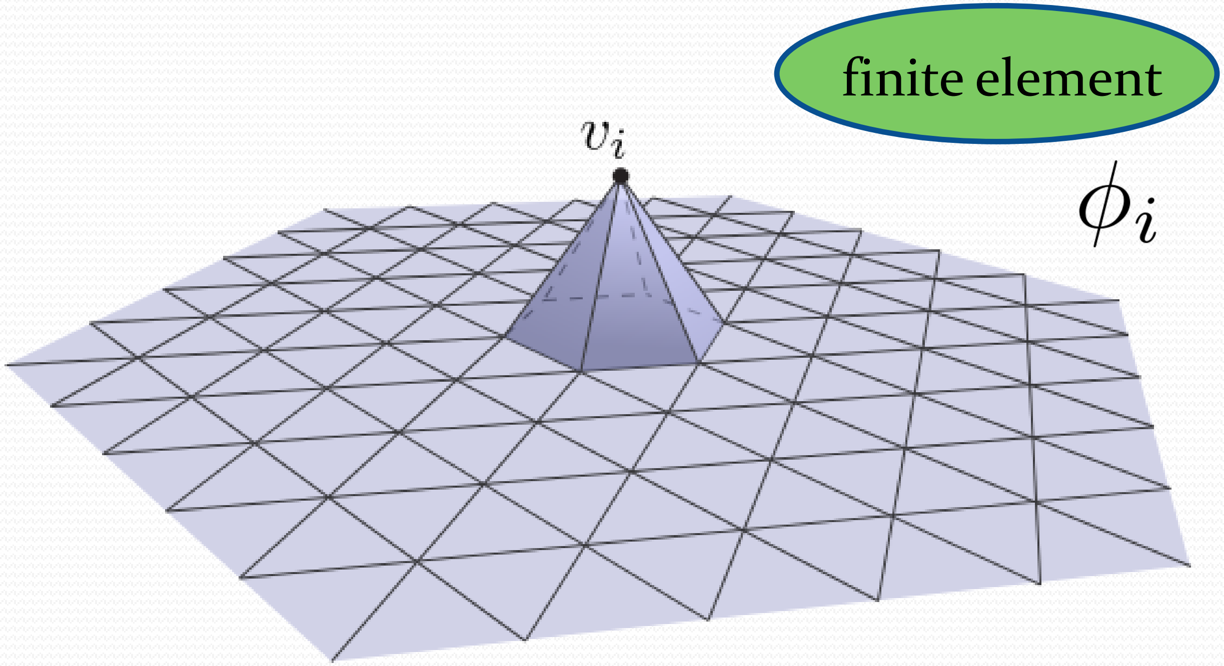
Simplest idea:
Function behaves linearly inside each triangle



Finite elements

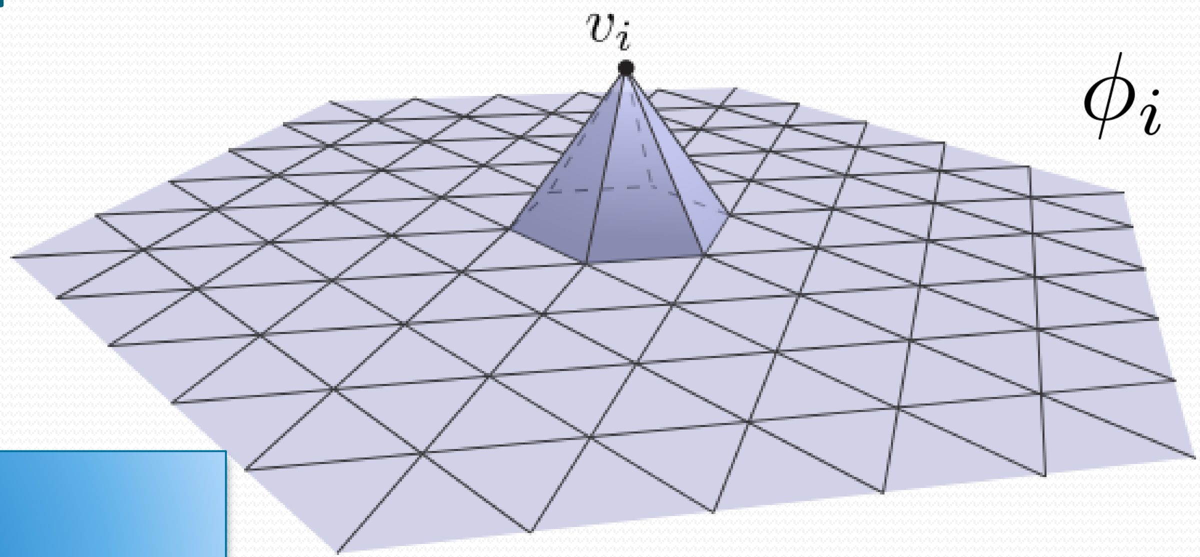
We want to construct a **piecewise linear** function, that fulfills $f(v_i) = f_i$ for given values f_i.

$$f(x) = \sum_i f_i \phi_i(x)$$



Discrete Laplacian

$$f(x) = \sum_i f_i \phi_i(x)$$



Rest of the lecture:
Derive

$$h(x) := \Delta f(x) = \sum_i h_i \phi_i(x)$$

We will see that

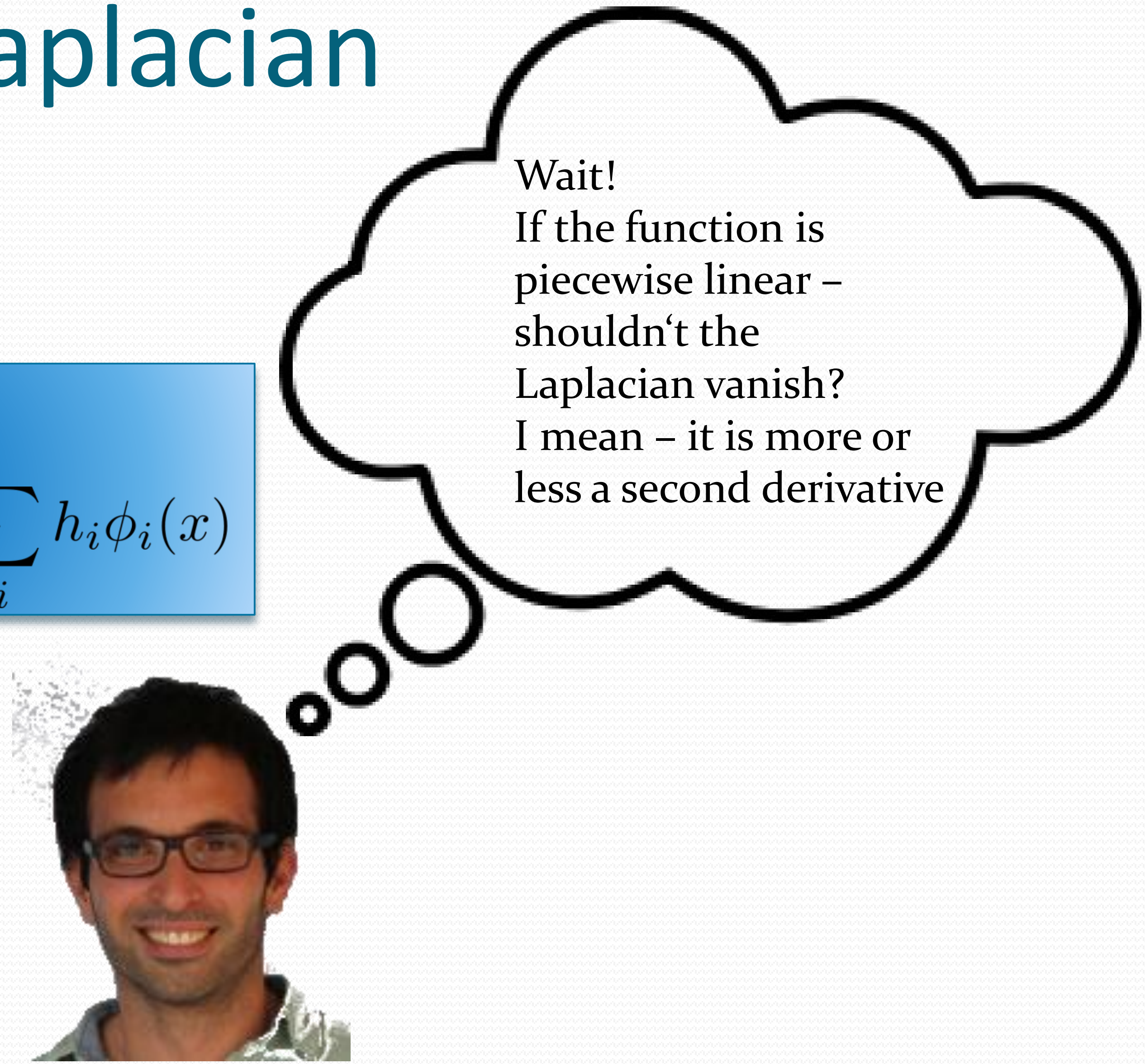
$$h = L f$$

Discrete Laplacian

$$f(x) = \sum_i f_i \phi_i(x)$$

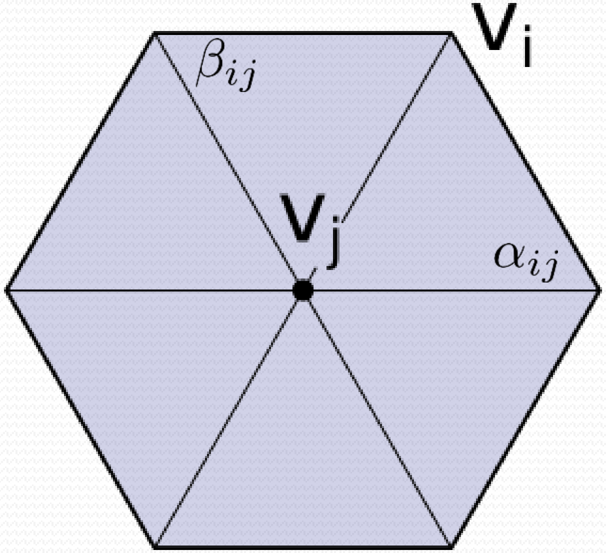
Rest of the lecture:
Derive

$$h(x) := \Delta f(x) = \sum_i h_i \phi_i(x)$$



Weak formulation

Let us multiply the equation with one of the basis functions and see, what happens:



$$h = \Delta f$$

$$\langle h, \phi_j \rangle = \langle \Delta f, \phi_j \rangle$$

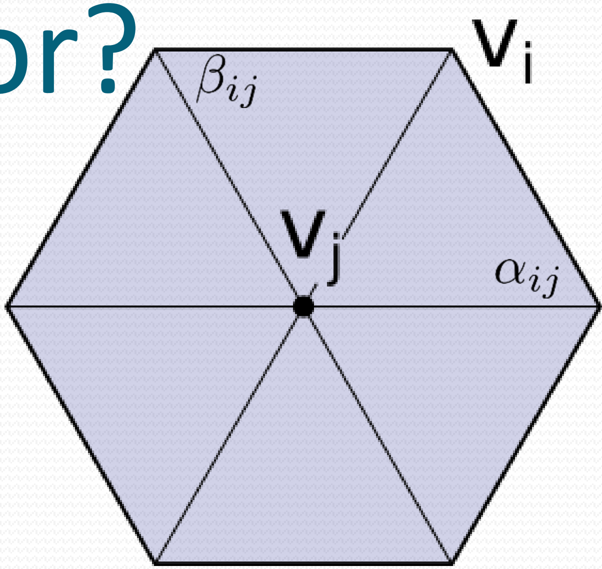
$$= -\langle \nabla f, \nabla \phi_j \rangle = -\sum_i f_i \underbrace{\langle \nabla \phi_i, \nabla \phi_j \rangle}_{-C_{ij}} = (Cf)_j$$

Integration by parts / Green's identity

Exercise:
Show that

$$= \frac{1}{2} \sum_{i \in N(j)} (\cot \alpha_{ij} + \cot \beta_{ij})(f_i - f_j)$$

What are we looking for?



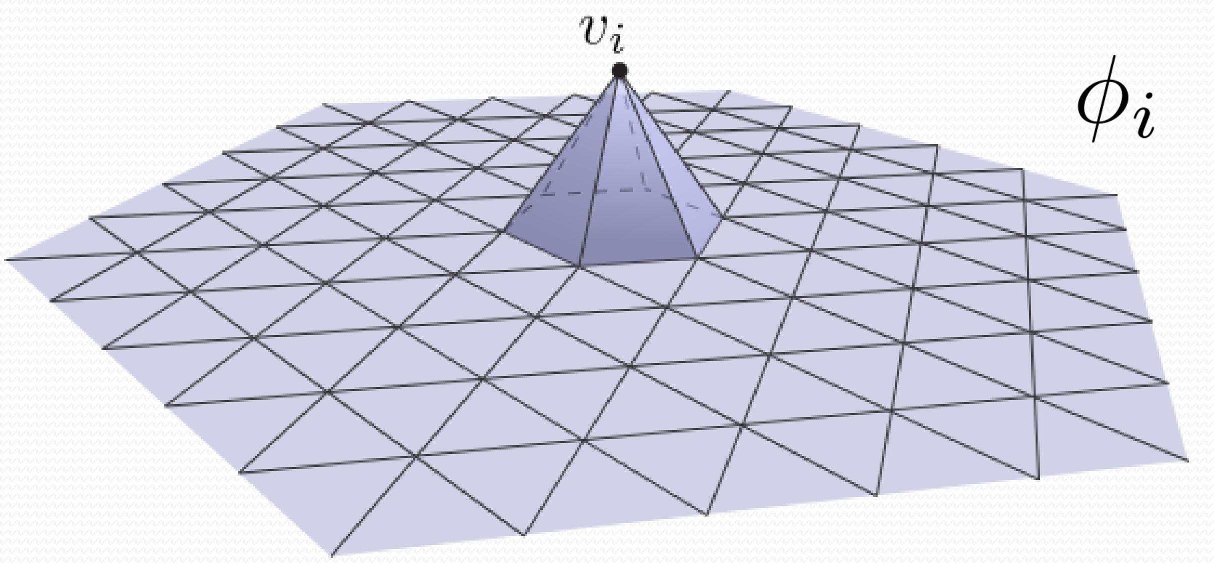
$$\langle h, \phi_j \rangle = \langle \Delta f, \phi_j \rangle$$

$$\langle \Delta f, \phi_j \rangle = \frac{1}{2} \sum_{i \in N(j)} (\cot \alpha_{ij} + \cot \beta_{ij})(f_i - f_j)$$

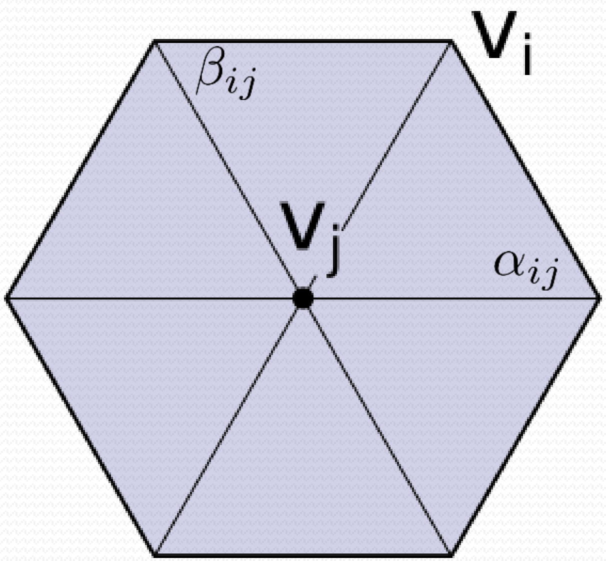
Remember, that we are looking for the coefficients of *h*, i.e. $h(v_i)$.

$$h(x) = \sum_i h_i \phi_i(x)$$

$$\cancel{\langle h, \phi_j \rangle = h_j} \quad ?$$



Evaluate 2nd term

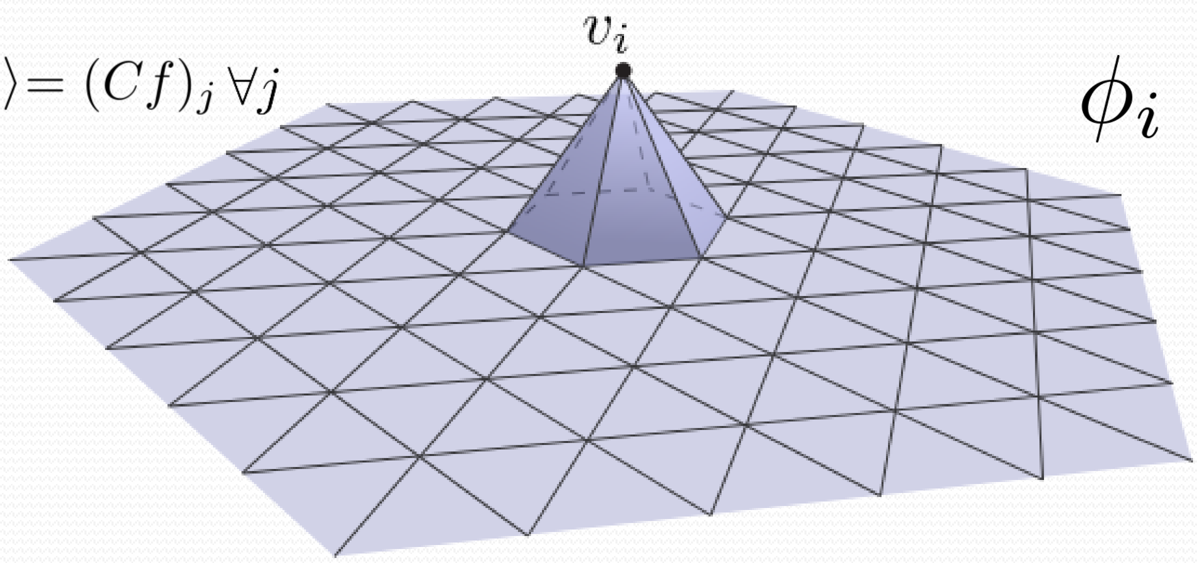


$$h(x) = \sum_i h_i \phi_i(x)$$

$$\langle h, \phi_j \rangle = \sum_i h_i \underbrace{\int \phi_i(x)\phi_j(x)dx}_{M_{ij}}$$

$$= (Mh)_j$$

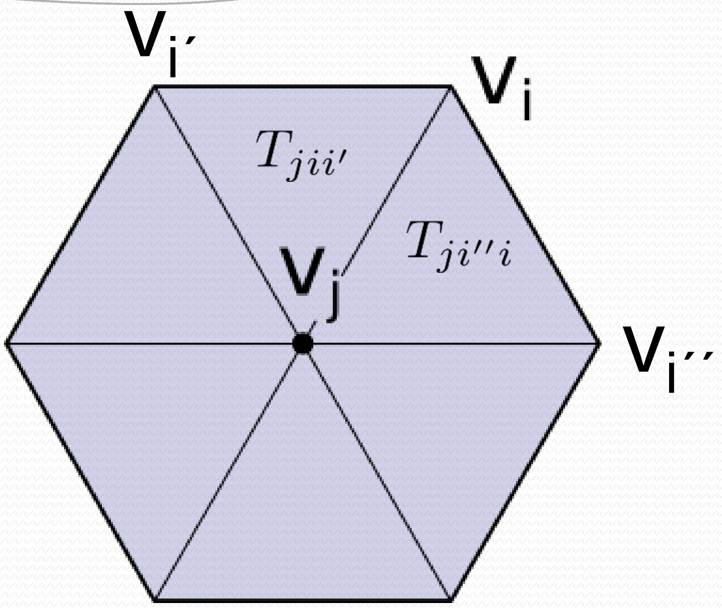
$$= -\sum_i f_i \langle \nabla\phi_i, \nabla\phi_j \rangle = (Cf)_j \,\forall j$$



This means:

$$h = M^{-1}Cf = Lf$$

The mass matrix



The integrals over the whole surface can be written as a sum of integrals over the triangles:

$$M_{ij} = \sum_{t \in \mathcal{T}} \int_T \phi_i(x)\phi_j(x)dx$$

The integrand is zero whenever the edge $\{v_i, v_j\}$ is not an edge of the triangle *T*. Thus only two summands remain:

$$M_{ij} = \int_{T_{jii'}} \phi_i(x)\phi_j(x)dx + \int_{T_{ji''i}} \phi_i(x)\phi_j(x)dx$$

We will now calculate the first integral.

The mass matrix

We want to calculate

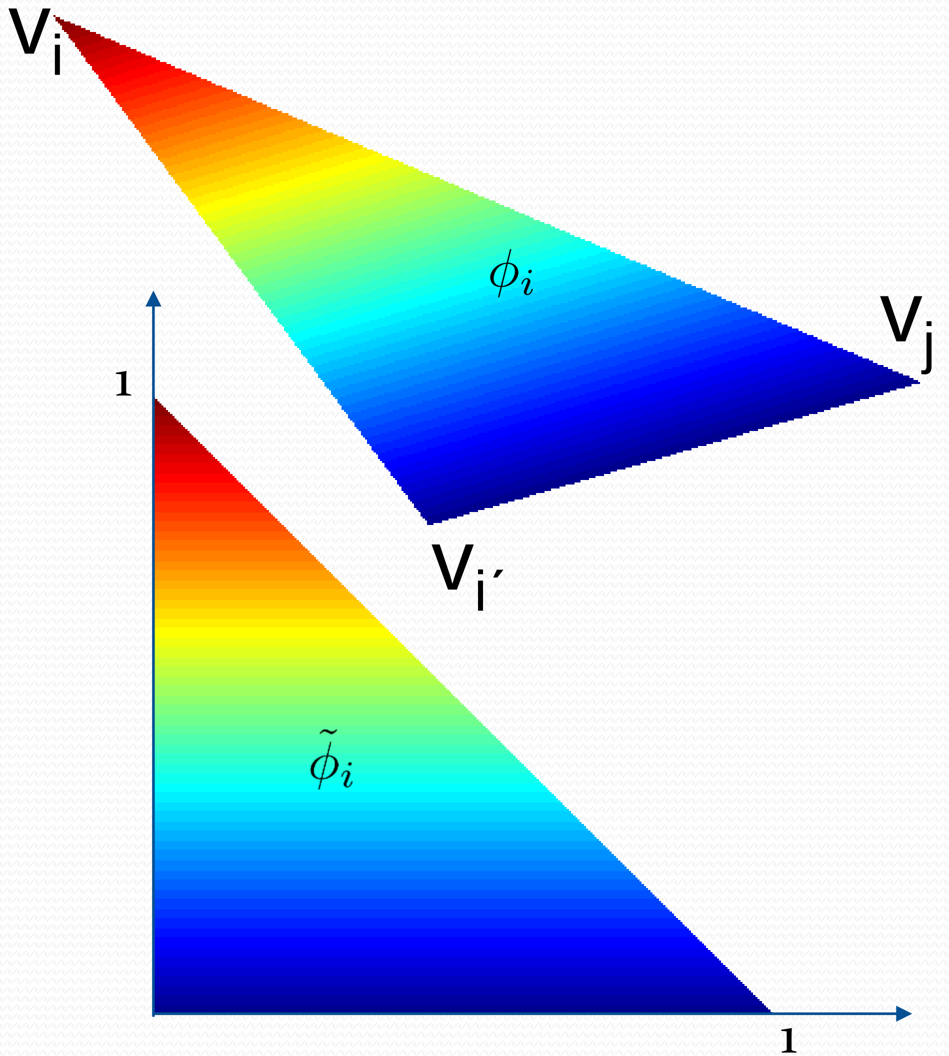
$$\int_{T_{jii'}} \phi_i(x)\phi_j(x)dx$$

for adjecent vertices v_i and v_j. This can be done by integrating in the parameter domain which in our case is the *reference triangle*:

$$\int_{T_{\text{ref}}} \underbrace{\tilde{\phi}_i(x)}_{x_1} \underbrace{\tilde{\phi}_j(x)}_{x_2} \underbrace{\sqrt{\det g}}_{2|T_{jii'}|}\, dx$$

$$= 2\,|T_{jii'}| \int_0^1 \int_0^{1-x_2} x_1 x_2 \, dx_1 dx_2$$

$$= \ldots = \frac{|T_{jii'}|}{12}$$



The mass matrix

For the diagonal elements the support of the integrand covers all triangles in the neighborhood of v_j:

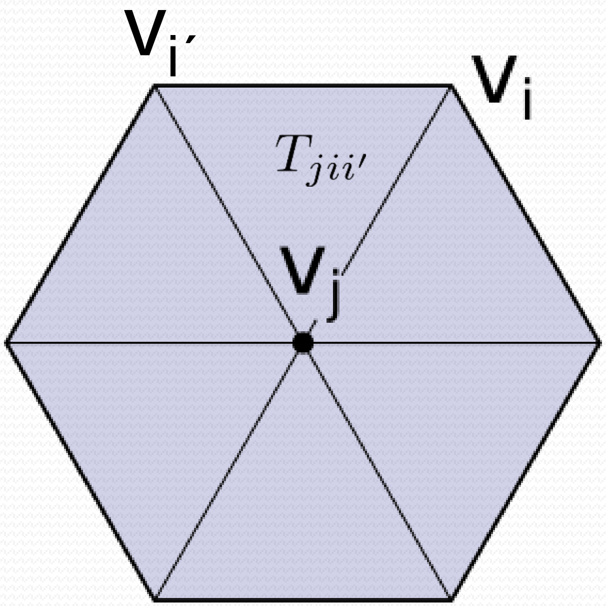
$$M_{jj} = \sum_{t \in \mathcal{T}} \int_T \phi_j(x)\phi_j(x)dx$$

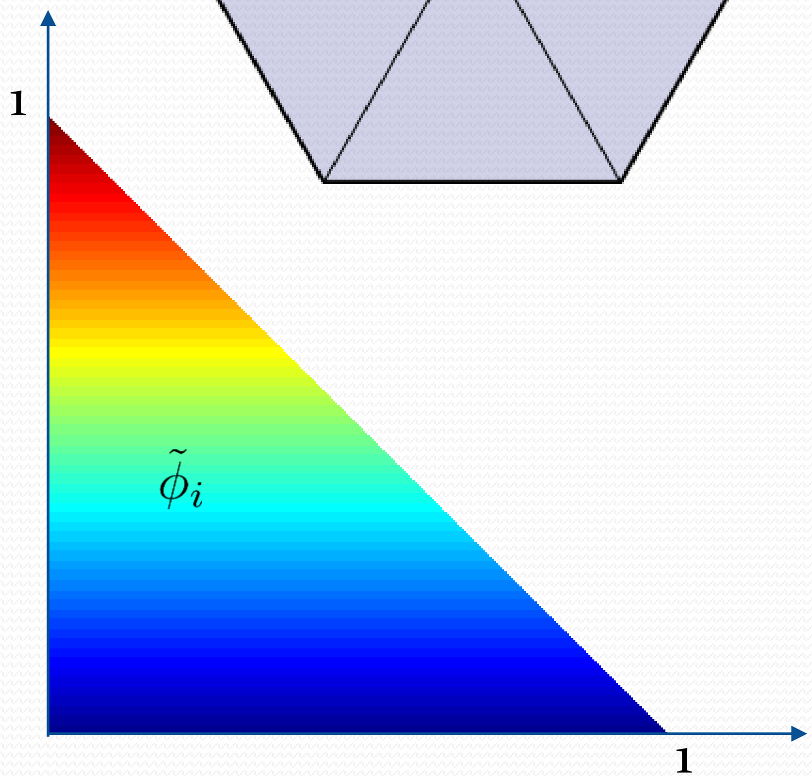
$$= \sum_{i \in N(j)} \int_{T_{jii'}} \phi_j(x)\phi_j(x)dx$$

Again we evaluate the integrals in the reference triangle:

$$\int_{T_{jii'}} \phi_j(x)^2 dx = 2\,|T_{jii'}| \int_0^1 \int_0^{1-x_2} x_2^2 \, dx_1 dx_2$$

$$= \ldots = \frac{|T_{jii'}|}{6}$$





Suggested reading

- http://brickisland.net/cs177/?p=309
- http://wwwmath.uni-muenster.de/num/Vorlesungen/WissenschaftlichesRechnen_WS1213/Dateien/Fem-intro.pdf