Analysis of Three-Dimensional Shapes
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Exercise Sheet 1

Room: 02.09.023

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Mathematics: Recap of Linear Algebra

Let us start with some definitions

Definition (Inner product). *Let X be a vector space. A mapping $\langle\cdot,\cdot\rangle : X \times X \to \mathbb{C}$ is called inner product, if*

1. $\langle x_1 + \lambda x_2, y\rangle = \langle x_1, y\rangle + \lambda\langle x_2, y\rangle \quad \forall x_i, y \in X, \lambda \in \mathbb{C}$
2. $\langle x, y\rangle = \overline{\langle y, x\rangle} \quad \forall x, y \in X$
3. $\langle x, x\rangle \geq 0 \quad \forall x \in X$
4. $\langle x, x\rangle = 0 \Leftrightarrow x = 0$

Two elements $x, y \in X$ are called perpendicular, if $\langle x, y\rangle = 0$.

Definition (Linear operator). *Let X and Y be vector spaces. A mapping $T : X \to Y$ is called linear, if*

$$T(x_1 + \lambda x_2) = T(x_1) + \lambda T(x_2)$$

A common notation is $Tx := T(x)$

Definition (Eigenvalues and eigenvectors). *Let $T : X \to X$ be a linear operator from a vector space X into itself (an endomorphism). An* eigenvector *is an element $0 \neq x \in X$ for which there exists a scalar $\lambda \in \mathbb{C}$, such that*

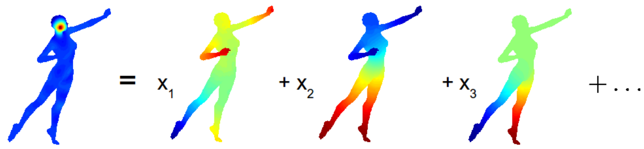
$$Tx = \lambda x$$

The scalar λ is called eigenvalue.

Exercise 1 (1 point). *1. Show that every matrix $\Phi \in \mathbb{C}^{n\times n}$ is representing an endomorphism on $\mathbb{C}^n$ via*

$$(\Phi x)_i = \sum_{j=1}^{n} \phi_{ij} x_j \quad \forall i = 1 \dots n$$

If we denote the j-th coloumn of Φ by ϕ_j we can write $\Phi x = \sum_{j=1}^{n} \phi_j x_j$



2. *Calculate the gradient $\nabla f(x) = (\partial_1 f(x), \dots, \partial_n f(x))^T$ for $f(x) = x^T A x$ ($A \in \mathbb{C}^{n\times n}$).*

Exercise 2 (1 point). *When not mentioned otherwise we consider the standard inner product on $\mathbb{C}^n$ given by*

$$\langle x, y \rangle = x^T \cdot \bar{y}$$

1. *Show that this is indeed an inner product.*
2. *Given a matrix $A \in \mathbb{C}^{n\times n}$, find the matrix B such that*

$$\forall x, y \in \mathbb{C}^n : \langle Ax, y \rangle = \langle x, By \rangle.$$

3. *Show that if $A = B$ then all the eigenvalues are real.*
4. *Show that if $A = B$ and the two eigenvectors x^1 and x^2 are not orthogonal, it follows $\lambda_1 = \lambda_2$.*

Programming: Working with Matlab and triangle meshes

A *triangle mesh* $\mathcal{M} = (\mathcal{V}, \mathcal{F})$ is a discrete surface embedded into $\mathbb{R}^3$. It consists of a *vertex* set $\mathcal{V} = \{v_1, \dots, v_n\}$, and a set of *triangles* $\mathcal{F} \subset \mathcal{V} \times \mathcal{V} \times \mathcal{V}$ (also called *faces* in a more general setting). The coordinates of the vertices embedded in $\mathbb{R}^3$ are denoted by $\mathbf{x}(v_1), \dots, \mathbf{x}(v_n) \in \mathbb{R}^3$. Note that the vertices of a triangle $t = (u, v, w) \in \mathcal{F}$ are ordered. We define triangles to be identical if they can be transformed into each other by a cyclic permutation, i.e. $(u, v, w) = (v, w, u) = (w, u, v)$ but $(u, v, w) \neq (w, v, u)$. This programming exercises will introduce you to working with meshes in Matlab.

Exercise 3 (One point for 1./2. and one point for 3./4./5.). *Download and expand the file `exercise1.zip` from the lecture website. Modifiy the files `adjacency.m`, `incidence.m`, `facearea.m`, `meshvolume.m` and `gaussiancurvature.m` to implement the functions as explained below. You can run the script `exercise.m` to test and visualize your solutions.*

1. *Given a triangle mesh* $\mathcal{M} = (\mathcal{V}, \mathcal{F})$ *with* n *vertices the* adjacency matrix $A \in \mathbb{R}^{n \times n}$ *is defined by*

$$A_{i,j} = \begin{cases} 1 & \exists t \in \mathcal{F} : v_i, v_j \in t \\ 0 & \textit{otherwise.} \end{cases}$$

   Implement function adjacency *that returns the adjacency matrix in sparse format for a given triangle mesh.*

2. *A mesh* $\mathcal{M} = (\mathcal{V}, \mathcal{F})$ *with* n *vertices induces a set of halfedges* $\mathcal{H} = \{(u, v) \in \mathcal{V} \times \mathcal{V} : \exists w \in \mathcal{V} : (u, v, w) \in \mathcal{F}\}$. *We assume some ordering on the halfedges, i.e.* $\mathcal{H} = \{h_1, \ldots, h_m\}$. *The* vertex-to-halfedge incidence matrix $I \in \mathbb{R}^{m \times n}$ *is defined by*

$$I_{i,j} = \begin{cases} -1 & h_i = (v_j, \cdot) \\ 1 & h_i = (\cdot, v_j) \\ 0 & \textit{otherwise.} \end{cases}$$

   Implement function incidence *that returns the vertex-to-halfedge incidence matrix in sparse format for a given triangle mesh.*

3. *Implement function* facearea *that returns the area of each triangle as an* $\mathbb{R}^{|\mathcal{F}|}$ *vector for a given triangle mesh.*

4. *Implement function* meshvolume *that returns the volume of a given triangle mesh. Hint: Think about the volume of a tetrahedron constructed from the vertices of a mesh triangle and the origin.*

5. *The* Gaussian curvature *at vertex* v *is given by* $\kappa_v = 2\pi - \sum_{t \in \mathcal{F}: v \in t} \theta_{t,v}$, *where* $\theta_{t,v}$ *is the angle of triangle* t *at vertex* v. *Implement function* gaussiancurvature *that returns the Gaussian curvature of each vertex as an* $\mathbb{R}^{|\mathcal{V}|}$ *vector for a given triangle mesh.*