Multiple View Geometry: Solution Exercise Sheet 10

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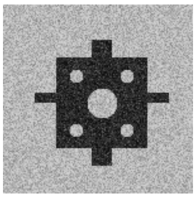
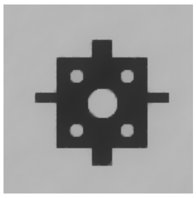
`http://vision.in.tum.de/teaching/ss2014/mvg2014`

Part I: Theory

1. The goal of image smoothing is to obtain the true image u from a corrupted image f.



Noisy image f



Smoothed image u

2. (a) In functional (1) the norm of the gradient of u is squared, in contrast to the functional (2).
   (b) f is the noisy image and u is the searched smoothed image.
   (c) The first part is called data term and enforces u to be similar to f. The second term is the regularizer which enforces u to be smooth.

3. (a) The Euler-Lagrange Equation can be derived by means of the Gateaux-Derivative.

(b - 1) Euler-Lagrange equation for E_1:

$$\begin{aligned} \frac{dE}{du} &= \frac{\partial \mathcal{L}}{\partial u} - \text{div} \frac{\partial \mathcal{L}}{\partial \nabla u}, \qquad \text{with } \mathcal{L}(u, \nabla u) = \frac{\lambda}{2}(u-f)^2 + \frac{1}{2}|\nabla u|^2 \\ &= \frac{\lambda}{2} \cdot 2(u-f) - \text{div}(\frac{1}{2} \cdot 2 \cdot \nabla u) \\ &= \lambda(u-f) - \text{div}(\nabla u) \end{aligned}$$

(c - 1) Gradient descent update step for E_1:

$$\begin{aligned} u^{t+1} &= u^t - \text{ dt} \cdot \frac{dE}{du^t} \\ &= u^t + \text{ dt } (\lambda(f-u^t) + \text{div}(\nabla u^t)), \quad \text{with time step dt.} \end{aligned}$$

$$\text{div}(\nabla u) = \text{div} \begin{pmatrix} u_x \\ u_y \end{pmatrix} = u_{xx} + u_{yy} \quad \text{with second derivatives } u_{xx} \text{ and } u_{yy}.$$

(b - 2)

$$\begin{aligned} \nabla u &= \begin{pmatrix} u_x \\ u_y \end{pmatrix} \\ |\nabla u| &= \sqrt{u_x^2 + u_y^2} \\ |\nabla u|_\epsilon &= \sqrt{u_x^2 + u_y^2 + \epsilon^2} = \sqrt{|\nabla u|^2 + \epsilon^2} = \left(|\nabla u|^2 + \epsilon^2\right)^{\frac{1}{2}} \end{aligned}$$

Euler-Lagrange equation for E_2:

$$
\begin{aligned}
\frac{dE}{du} &= \frac{\partial \mathcal{L}}{\partial u} - \text{div}\frac{\partial \mathcal{L}}{\partial \nabla u}, \qquad \text{with } \mathcal{L}(u, \nabla u) = \frac{\lambda}{2}(u-f)^2 + |\nabla u|_\epsilon \\
&= \frac{\lambda}{2} \cdot 2(u-f) - \text{div}\left(\tfrac{1}{2}\left(|\nabla u|^2 + \epsilon^2\right)^{-\frac{1}{2}} \cdot 2 \cdot \nabla u\right) \\
&= \lambda(u-f) - \text{div}\left(\frac{\nabla u}{(|\nabla u|^2 + \epsilon^2)^{\frac{1}{2}}}\right) \\
&= \lambda(u-f) - \text{div}\left(\frac{\nabla u}{|\nabla u|_\epsilon}\right)
\end{aligned}
$$

(c - 2) Gradient descent update step for E_2:

$$
\begin{aligned}
u^{t+1} &= u^t - \text{ dt} \cdot \frac{dE}{du^t} \\
&= u^t + \text{ dt } \left(\lambda(f - u^t) + \text{div}\left(\frac{\nabla u^t}{|\nabla u^t|_\epsilon}\right)\right), \quad \text{with time step dt.}
\end{aligned}
$$