Analysis of Three-Dimensional Shapes
(IN2238, TU München, Summer 2015)

Regular Surfaces
(28.04.2015)

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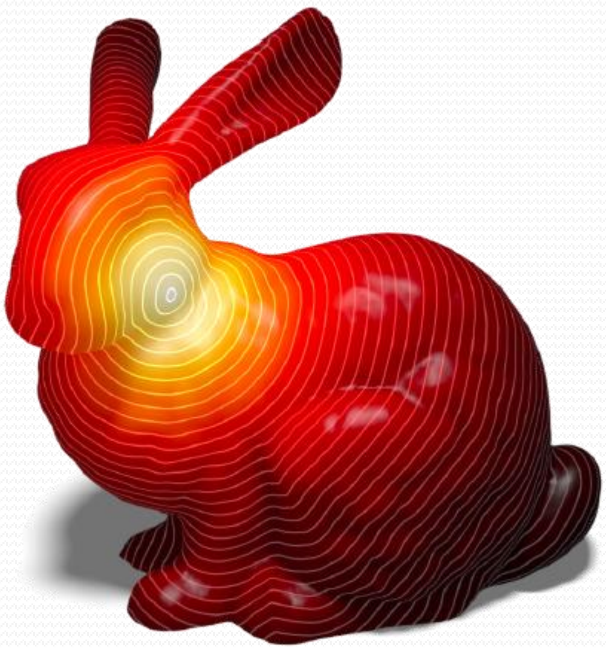
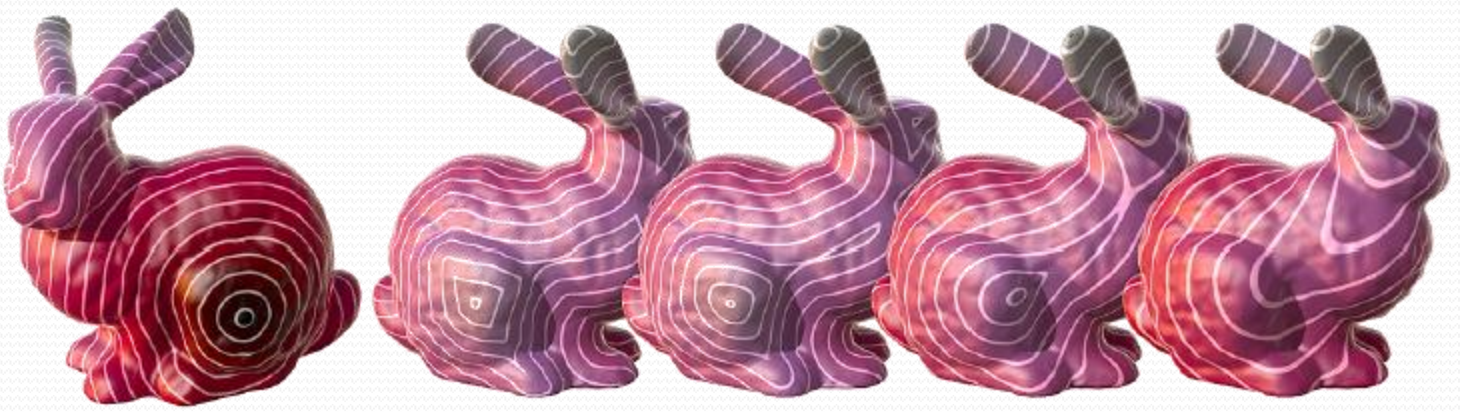
Room 02.09.058, Informatik IX

Seminar



"Geodesics in Heat"
Diana Papyan

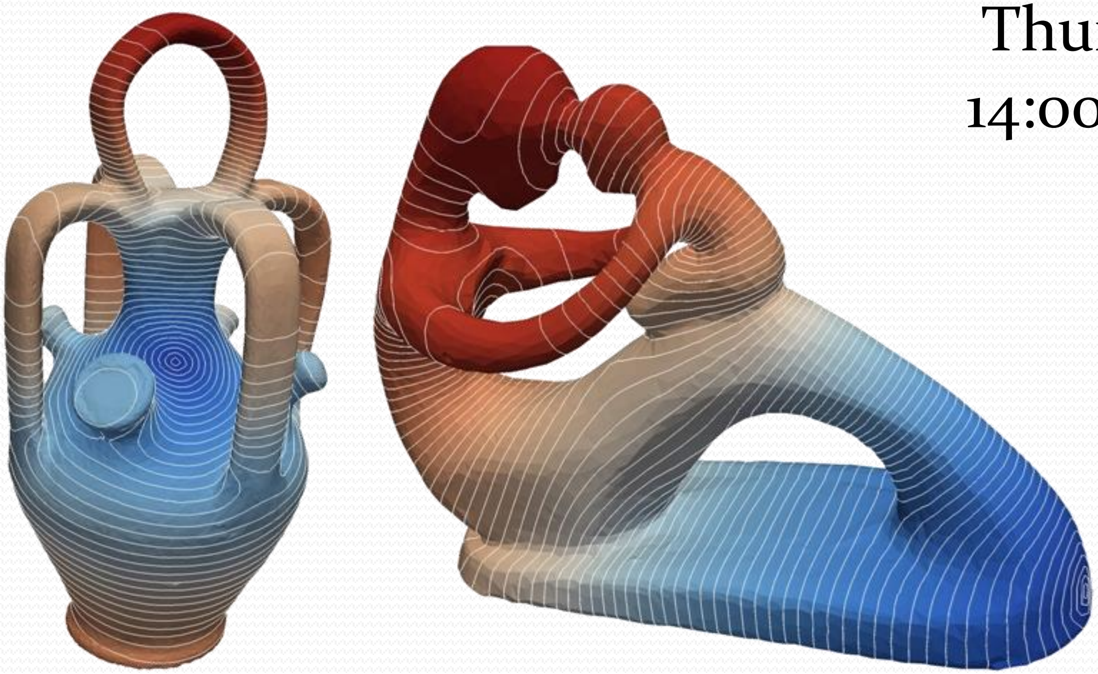
Thursday, April 30th
14:00 Room 02.09.023



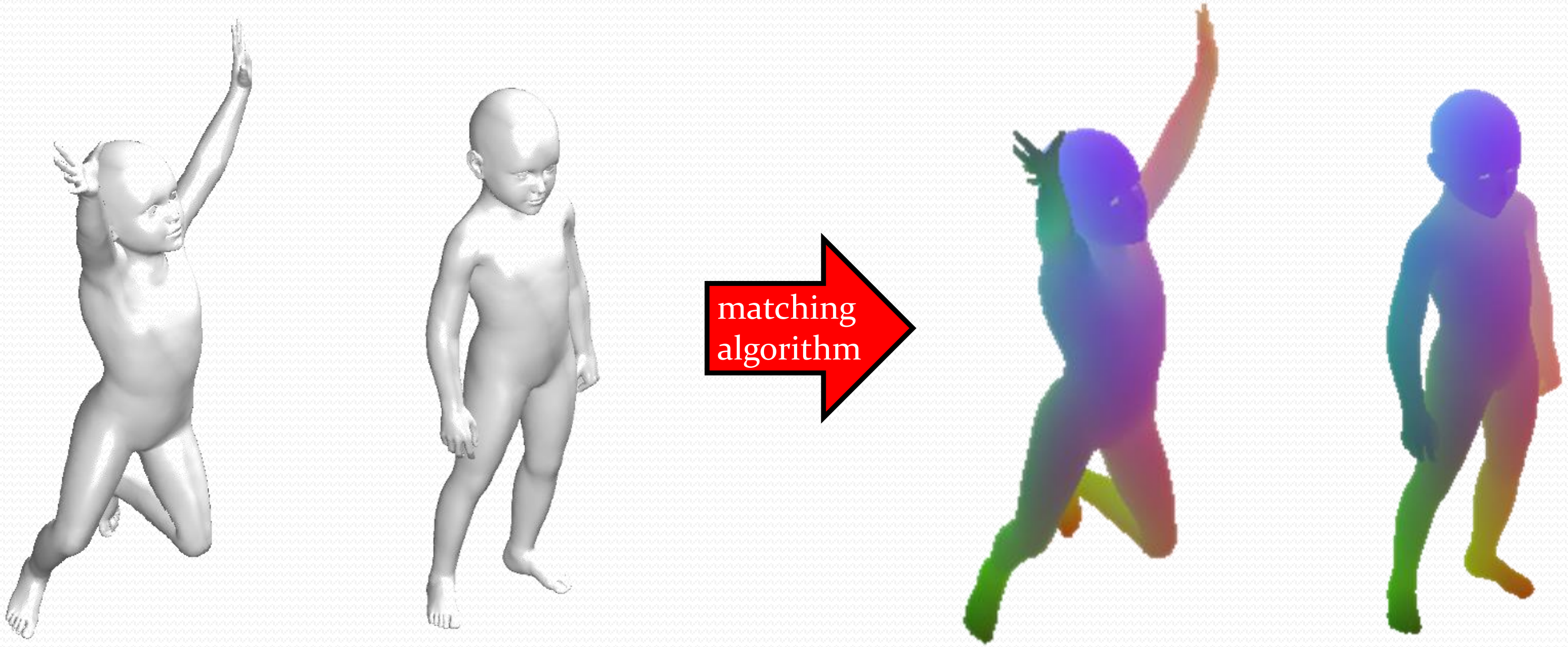
Seminar

"Biharmonic Distance"
Björn Häfner

Thursday, April 30th
14:00 Room 02.09.023



The matching game

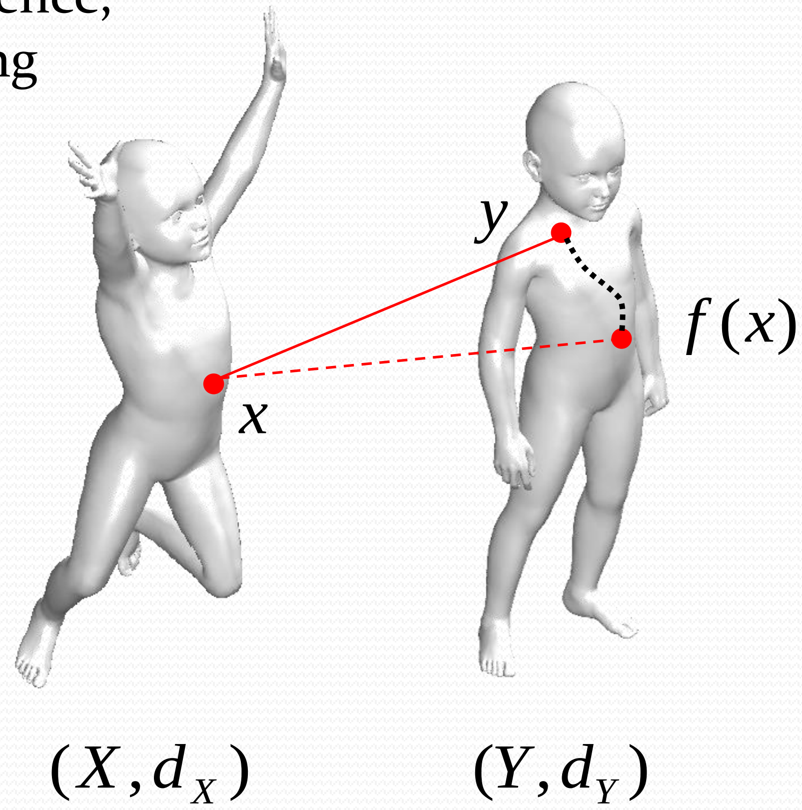


The matching game

Let $C \subset X \times Y$ be the computed correspondence, and $f : X \rightarrow Y$ be the ground-truth mapping among the two shapes (which we have).

The *average geodesic error* of C is defined by

$$\varepsilon(C) = \frac{1}{|C|} \sum_{(x,y)\in C} d_Y(y, f(x))$$



The matching game

Let A and B be the number of matched points in X and Y respectively, and let N be the total number of points.

We compute the *score* of C as:

$$\mathrm{score}(C) = \frac{A+B}{2N}\,\frac{1-\varepsilon(C)}{\mathrm{diam}\,Y}$$

$N = 59727$

$\mathrm{diam}\,Y = 119.83$

$A = 100\%$

$B = 36.64\%$

$\varepsilon = 8.81$

$\mathrm{score} = 0.633$

Overview

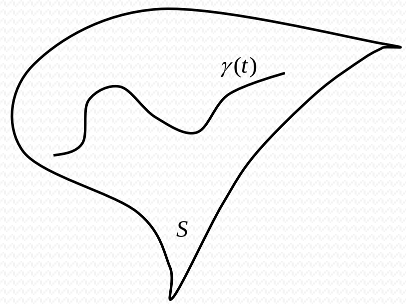
Differential geometry is the study of **local properties** of curves and surfaces, i.e. those properties which depend only on the behavior of the curve or surface in the neighborhood of a point.

Some of these local properties **act globally**, in the sense that they also have an influence on the behavior of the entire curve or surface (e.g. characterizing geodesic paths on a surface).

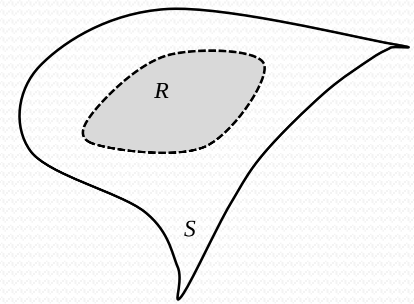
Common notions in Computer Vision and Graphics such as **curvature**, **normal vectors**, **geodesic distance**, **area elements** and so on are part of differential geometry.

Overview

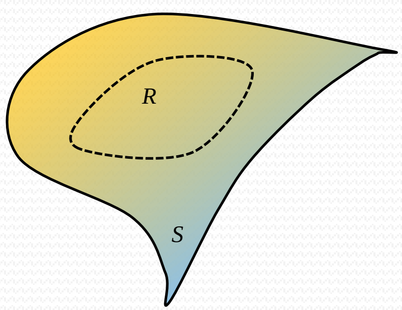
Differential geometry provides us with powerful tools to directly compute, among other things:



Length of a curve

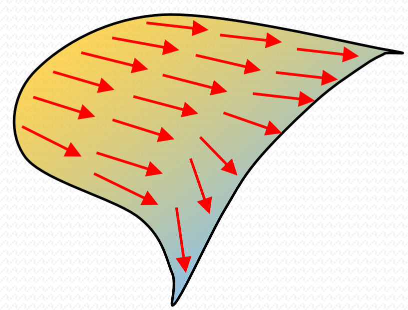


Area of a region



Integral of a function

$f : S \to \mathbf{R}$

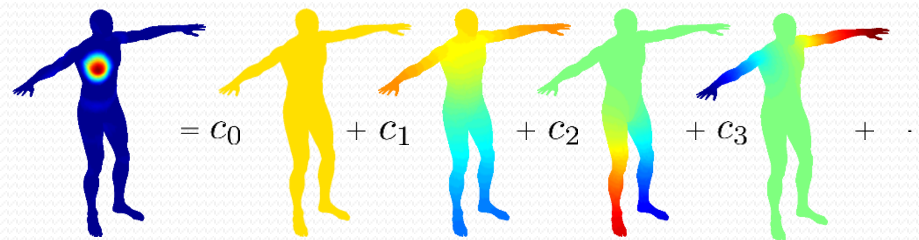


Gradient of a function

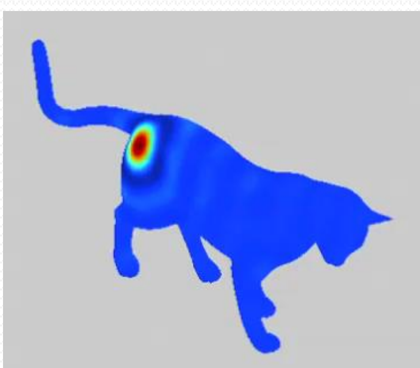
Overview

…which in turn enables us to do some more interesting and fancy stuff, like:

Texture segmentation



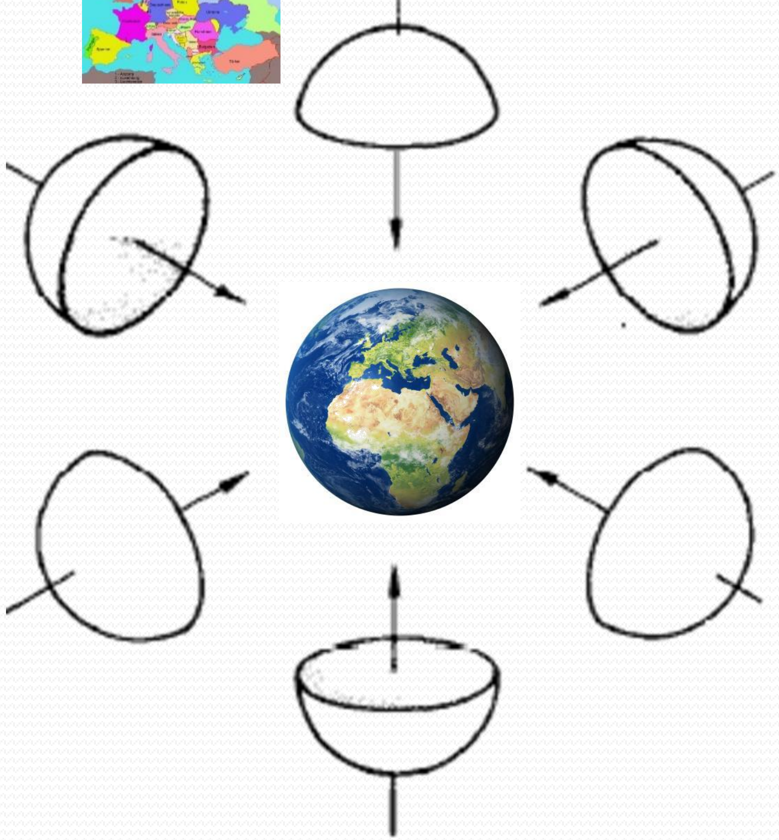
Fourier analysis



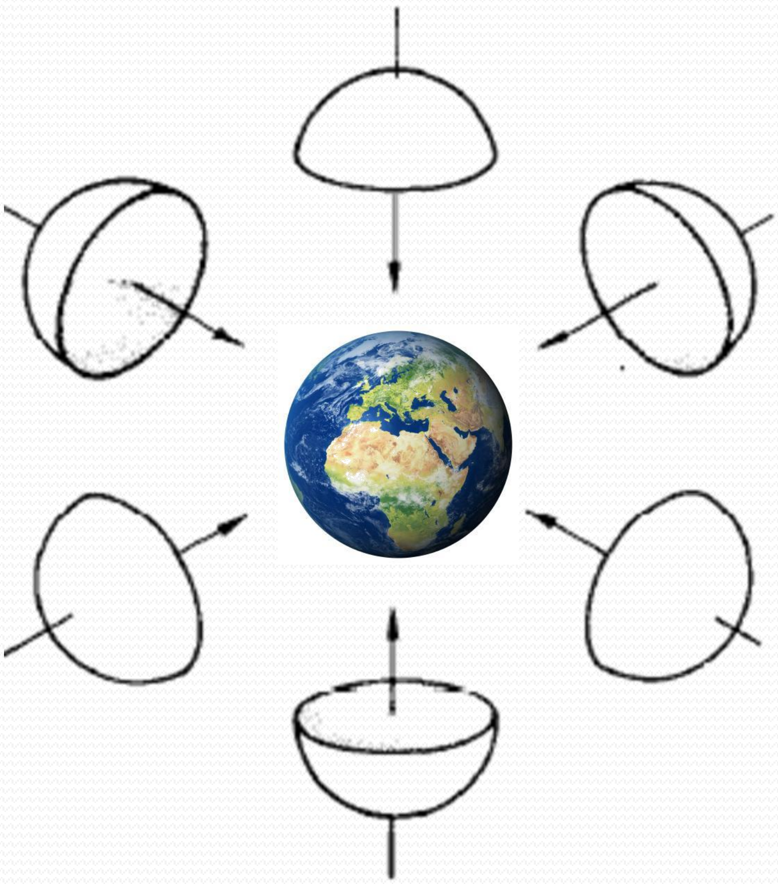
Solving differential equations

Shape matching

This is your world



A union of charts



Each of these charts can be seen as a mapping
$\mathbf{x} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$

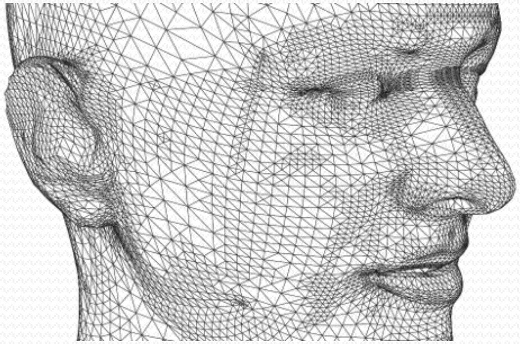
We already know that we cannot require **x** to be an **isometry** (see the cartographer's example from yesterday).

But we can require **x** to be smooth and invertible, in particular **x** should be a **diffeomorphism** (see slides about Lipschitz distance).

Regular surfaces

Intuitively: A **regular surface** in $\mathbf{R}^3$ is obtained by taking pieces of a plane, deforming them, and arranging them so that the resulting shape has no sharp points, edges, or self-intersections.

This way, it makes sense to speak of *tangent planes*, and the figure is *smooth* enough so that the usual notions of calculus can be extended to it.



Manifolds

From the point of view of classical differential geometry, a regular surface is a **2-dimensional Riemannian sub-manifold**.

it is a union of 2D charts

it has a *metric* structure (we will see this next week)

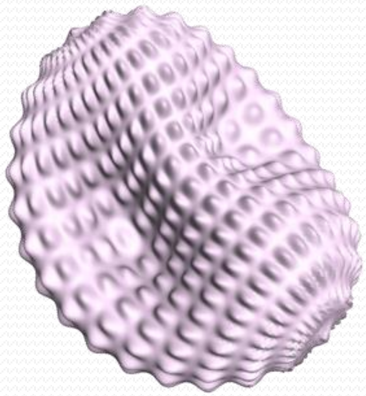
subset of $\mathbb{R}^3$

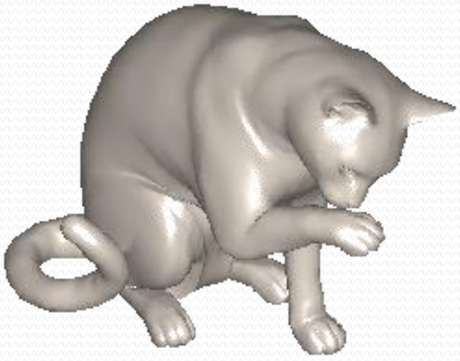
With this in mind, from now on whenever we refer to 2D manifolds we will mean «regular surface».

Most papers on shape analysis use these terms interchangeably.

Manifolds without boundary

For the rest of the lecture we will consider *regular surfaces without boundary*.





Note that this is but one particular choice. For example, we could instead model our shapes as 3-dimensional manifolds *with* boundary (interior+surface).

Parametrized curves

A **parametrized differentiable curve** is a differentiable map $\alpha : I \to \mathbf{R}^3$ of an open interval $I = (a,b)$ of the real line into $\mathbf{R}^3$.

$$\alpha(t) = (x(t), y(t), z(t))$$

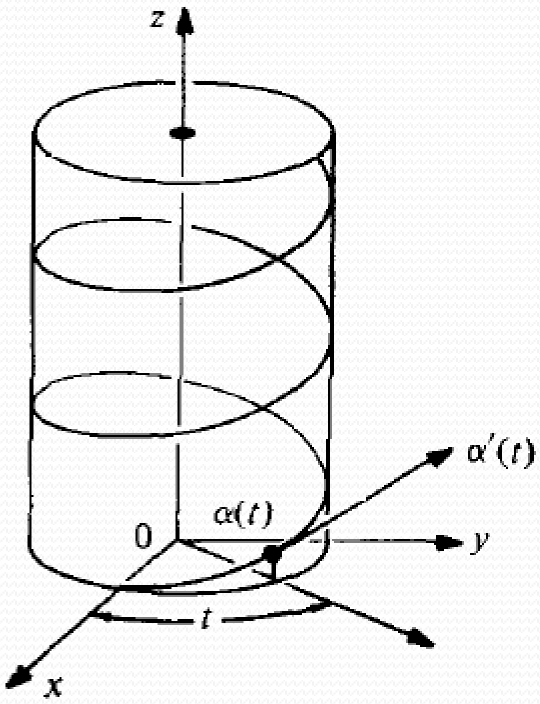
- t is called *parameter*
- $x(t)$, $y(t)$, $z(t)$ are differentiable

The **tangent vector** (or velocity vector) of the curve at t is defined as:

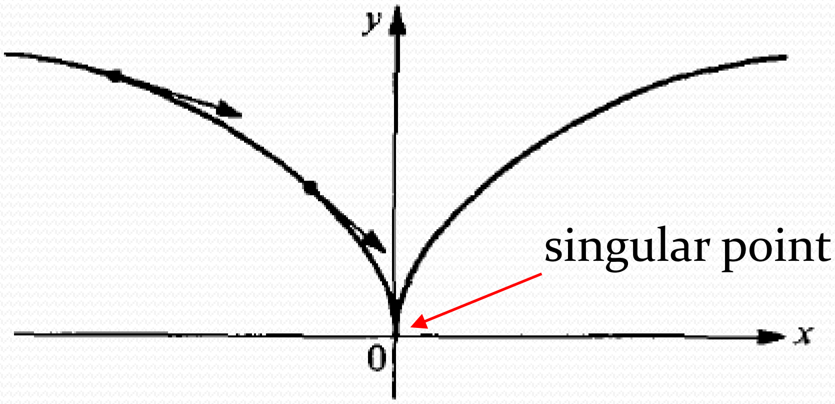
$$\alpha'(t) = (x'(t), y'(t), z'(t))$$

Parametrized curves

$\alpha(t) = (a\cos(t), a\sin(t), bt)$



$\alpha(t) = (t^3, t^2)$



Both curves are differentiable, the second curve has $\alpha'(0) = (0,0)$, thus only the first curve is **regular**.

Parametrized surfaces

A parametrized surface element or **chart** is a **regular homeomorphism**

$$\mathbf{x} : U \to S \subset \mathbb{R}^3$$

Here $U \subset \mathbb{R}^2$ is open, and $\mathbf{x} : U \to S \subset \mathbb{R}^3$ is regular if $\mathrm{d}\mathbf{x}_p : U \to \mathbb{R}^3$ has full rank.

This means that the parametrization is differentiable:

$$\mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v)) \qquad u,v \in U$$

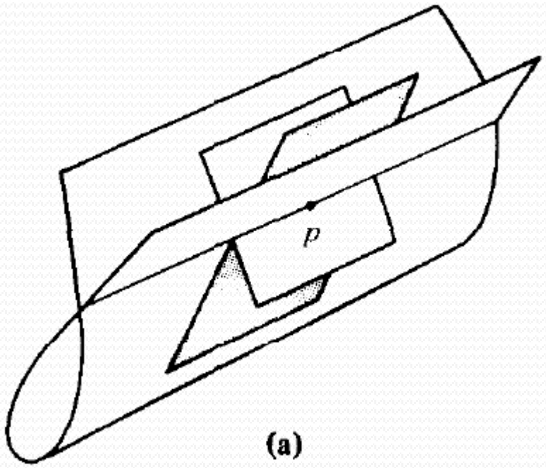
where functions x, y, z have continuous partial derivatives of all orders in U.

The full rank condition means that there exists a tangent plane at all points of S (we will come back to this later).

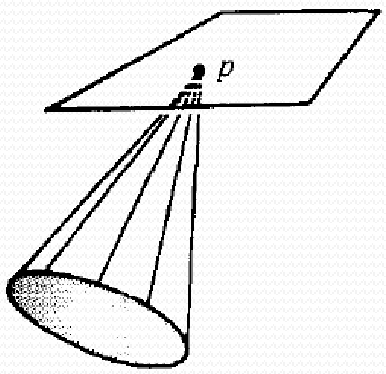
Parametrized surfaces

A parametrized surface (regular surface) is the union of parametrized surface elements (charts):

Non-regular surfaces



(a)

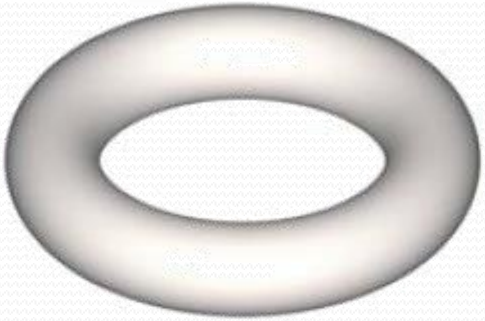


(b)

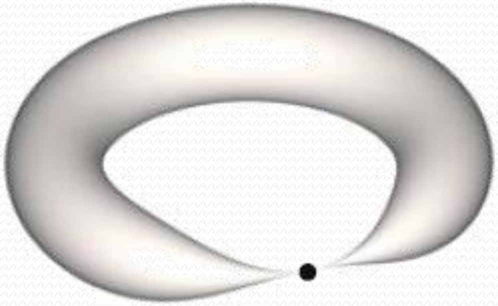
Self-intersection: since we require $\mathbf{x} : U \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$ to be a homeomorphism, it must be one-to-one. However, this is not the case at point *p* in the figure.

Cusps and edges: these are singular points in the same sense as we had with the regular curves. In the figure above, we cannot really speak about a tangent plane at *p*.

Some examples



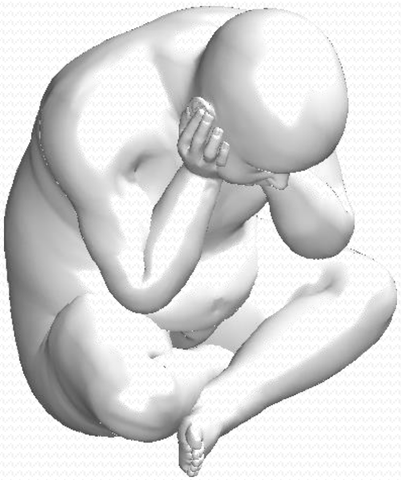
2-dimensional manifold *without* boundary



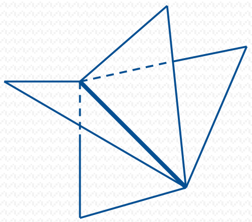
not a manifold (not differentiable at the kink)

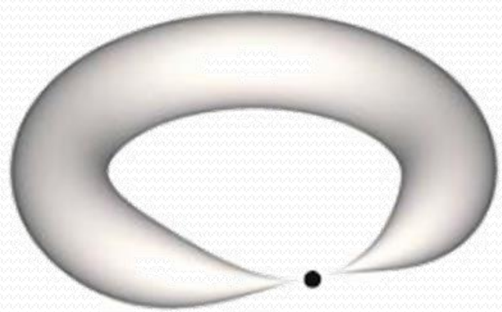
2-dimensional manifold *with* boundary (the boundary itself is a 1-dimensional manifold)

Non-manifolds

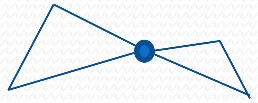


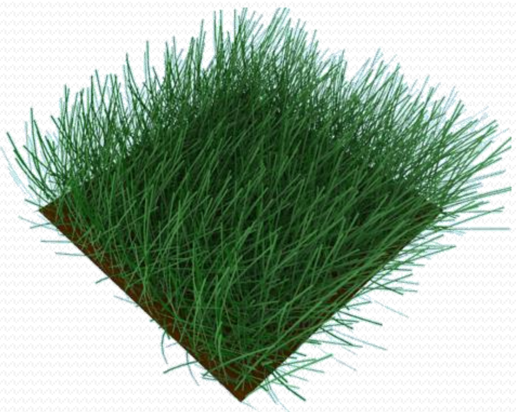
self-intersecting geometry (tangent plane is not unique)



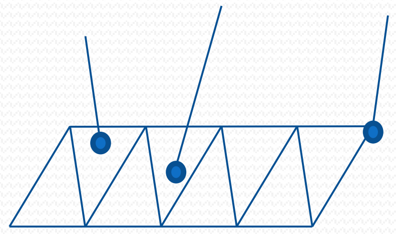


topological noise

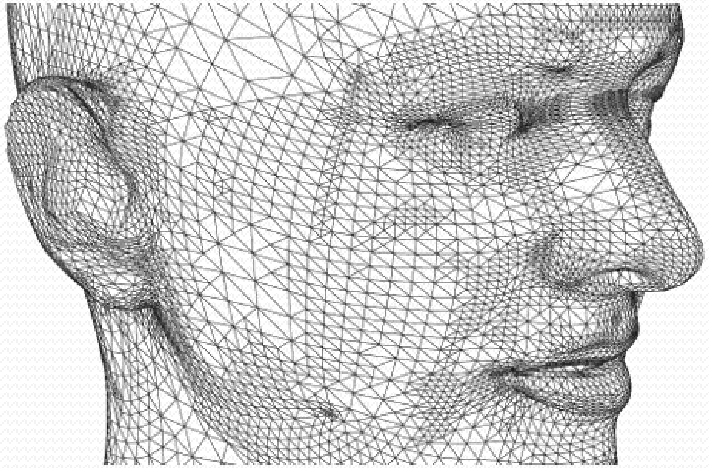




lower-dimensional structures



Discrete meshes



As a matter of fact, in practice **none** of our meshes are manifolds!

Meshes are defined as collections of polygons (usually triangles), hence we always have **irregularities** between adjacent faces and at the vertices.

However, we will still be able to define meaningful quantities which approximate well (in some sense) their continuous counterparts.

Discretizing the notions of differential geometry to work with meshes is the main task of **discrete differential geometry**.

Example of regular surface

Let us show that the unit sphere

$$S^2 = \{(x, y, z) \in \mathbf{R}^3; x^2 + y^2 + z^2 = 1\}$$

is a regular surface.

Consider the parametrization $\mathbf{x}_1 : U \subset \mathbf{R}^2 \to \mathbf{R}^3$ given by

$$\mathbf{x}_1(x, y) = \left(x, y, \sqrt{1-(x^2+y^2)}\right)$$

where $U = \{x, y \in \mathbf{R}^2; x^2 + y^2 < 1\}$.

$\mathbf{x}_1(U)$ is the open part of S^2 above the *xy* plane.

Example of regular surface

Since $x^2 + y^2 < 1$, the function $\sqrt{1-(x^2+y^2)}$ has continuous partial derivatives of all orders and thus $\mathbf{x}_1$ is differentiable.

Similarly, consider the parametrization

$$\mathbf{x}_2(x, y) = \left(x, y, -\sqrt{1-(x^2+y^2)}\right)$$

Observe that $\mathbf{x}_1(U) \cup \mathbf{x}_2(U)$ covers S^2 minus the equator:

Example of regular surface

We can proceed and define the additional parametrizations:

$$\mathbf{x}_3(x,z) = \left(x, \sqrt{1-(x^2+z^2)}, z\right)$$

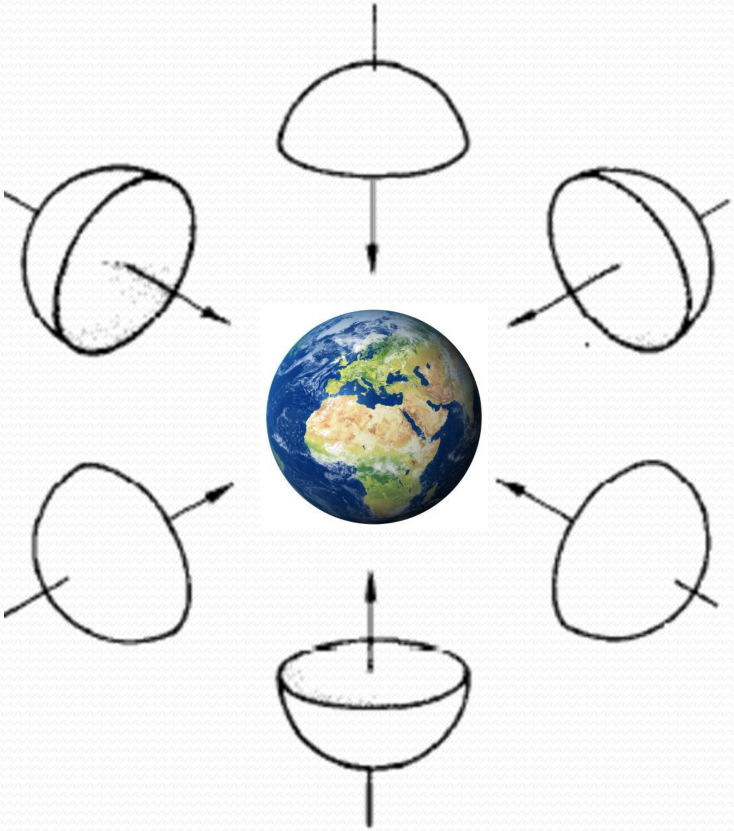
$$\mathbf{x}_4(x,z) = \left(x, -\sqrt{1-(x^2+z^2)}, z\right)$$

$$\mathbf{x}_5(y,z) = \left(\sqrt{1-(y^2+z^2)}, y, z\right)$$

$$\mathbf{x}_6(y,z) = \left(-\sqrt{1-(y^2+z^2)}, y, z\right)$$

These, together with $\mathbf{x}_1$ and $\mathbf{x}_2$, cover S^2 completely and show that it is indeed a regular surface.

Example of regular surface



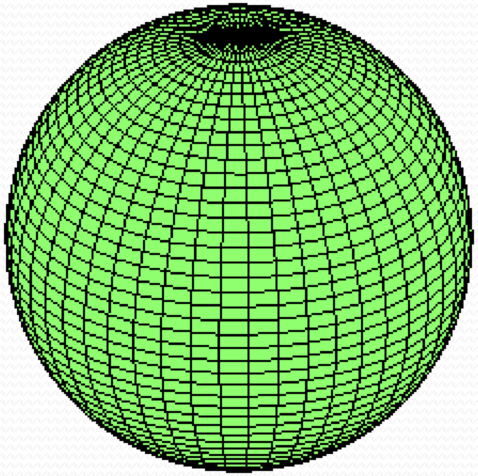
Examples

$$\mathbf{x} : (-5,5)^2 \to \mathbb{R}^3$$
$$\mathbf{x}(u,v) = (ua\cos(v), ua\sin(v), bv)$$

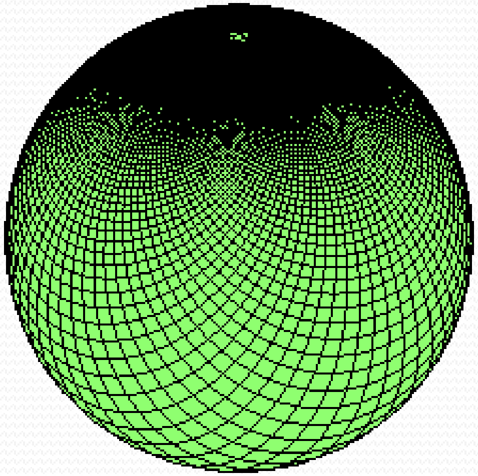
$$\mathbf{x} : (0,6\pi) \times (0,2\pi) \to \mathbb{R}^3$$
$$\mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v))$$
$$x(u,v) = 2(1 - e^{\frac{u}{6\pi}})\cos(u)\cos(\tfrac{v}{2})^2$$
$$y(u,v) = 2(e^{\frac{u}{6\pi}} - 1)\sin(u)\cos(\tfrac{v}{2})^2$$
$$z(u,v) = 1 - e^{\frac{u}{3\pi}} - \sin(v) + e^{\frac{u}{6\pi}}\sin(v)$$

Examples

$$\mathbf{x} : (0, 2\pi) \times (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \to \mathbb{R}^3$$
$$\mathbf{x}(u, v) = (\cos(u)\cos(v), \sin(u)\cos(v), \sin(v))$$



$$\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3$$
$$\mathbf{x}(u, v) = \tfrac{1}{u^2+v^2+1}(2u, 2v, u^2 + v^2 - 1)$$



Change of parameters

As also seen from the previous example (the world maps), in general a surface point *can belong to many surface elements / charts*.

In fact, in general we could choose other coordinate systems and parametrizations, for example via stereographic projection or geographical coordinates.

The local properties of the surface should not depend on the specific choice of a system of coordinates!

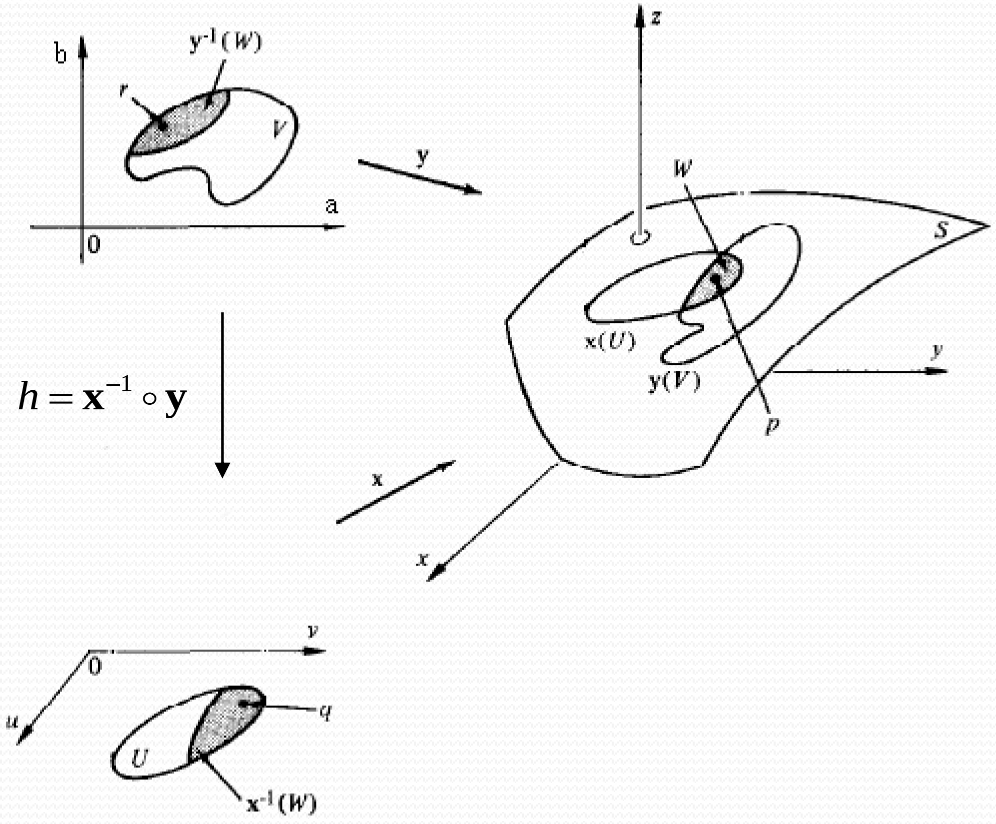
Change of parameters

The following proposition will be useful when we will have to prove invariance to parametrization:

Let p be a point of a regular surface S, and let $\mathbf{x}: U \subset \mathbf{R}^2 \to S$ and $\mathbf{y}: V \subset \mathbf{R}^2 \to S$ be two parametrizations of S such that $p \in \mathbf{x}(U) \cap \mathbf{y}(V) = W$. Then the *change of coordinates* $h = \mathbf{x}^{-1} \circ \mathbf{y} : \mathbf{y}^{-1}(W) \to \mathbf{x}^{-1}(W)$ is a diffeomorphism (that is, h is differentiable and has a differentiable inverse).

Simply put, if p belongs to two neighborhoods $\boldsymbol{x}(U)$ and $\boldsymbol{y}(V)$, it is possible to pass from one coordinate system to the other by means of a differentiable transformation.

Change of parameters



Differentiable function on a surface

We will now define the notion of a *differentiable function* on a regular surface.

Let $f : V \subset S \to \mathbf{R}$ be a function defined in an open subset *V* of a regular surface *S*. Then *f* is said to be **differentiable** at *p* if, for some parametrization $\mathbf{x} : U \subset \mathbf{R}^2 \to S$ with $\mathbf{x}(U) \subset V$, the composition $f \circ \mathbf{x} : U \subset \mathbf{R}^2 \to \mathbf{R}$ is differentiable at $\mathbf{x}^{-1}(p)$.

Thus, a function *f* is differentiable at *p* if its expression in the coordinate neighborhood spanned by (u,v) admits continuous partial derivatives of all orders.

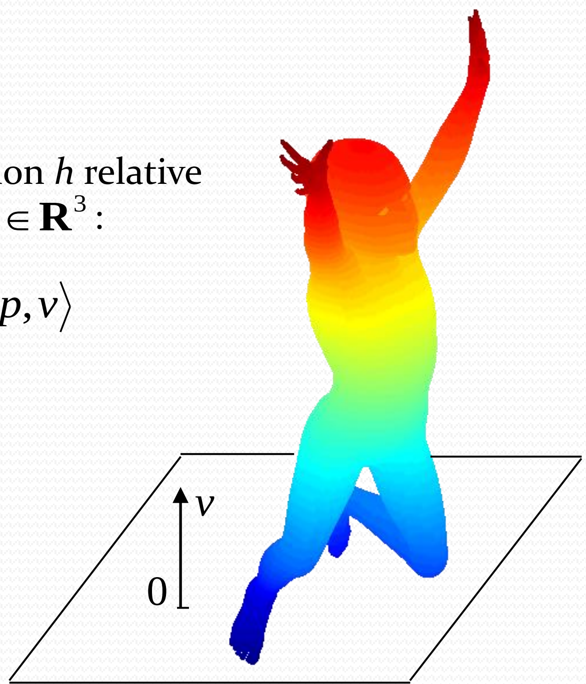
Differentiable function on a surface

Note that this definition *does not depend on the choice of the parametrization*. In fact, if $\mathbf{y}: U \subset \mathbf{R}^2 \to S$ is another parametrization with $p \in \mathbf{x}(V)$, and if $h = \mathbf{x}^{-1} \circ \mathbf{y}$, then $f \circ \mathbf{y} = f \circ \mathbf{x} \circ h$ is also differentiable.

<u>Example:</u>

The height function h relative to a unit vector $v \in \mathbf{R}^3$:

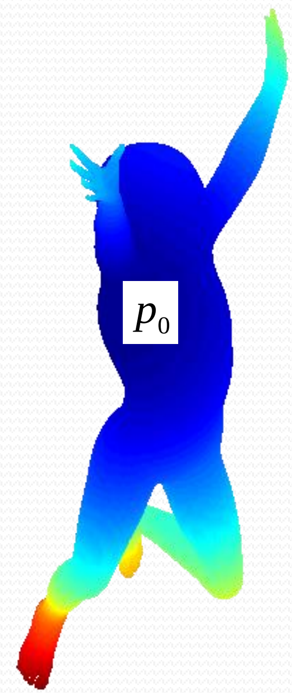
$$h(p) = \langle p, v \rangle$$



<u>Example:</u>

The distance function

$$d(p) = \| p - p_0 \|^2$$



Differentiable mappings

We have previously seen the notion of homeomorphic functions among shapes (for example, isometries). Similarly, we can extend the definition of differentiability to mappings between surfaces.

A continuous map $\varphi : V_1 \subset S_1 \to S_2$ is said to be differentiable at *p* if, given parametrizations

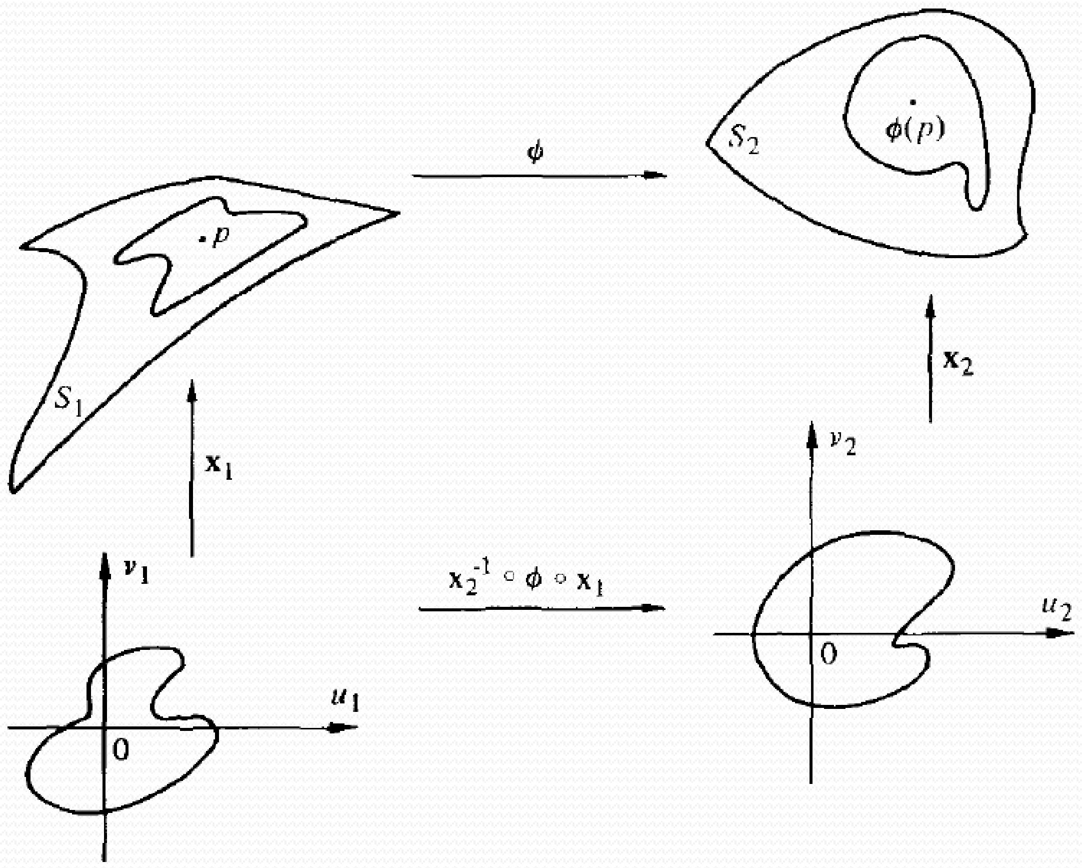
$$\mathbf{x}_1 : U_1 \subset \mathbf{R}^2 \to S_1, \qquad \mathbf{x}_2 : U_2 \subset \mathbf{R}^2 \to S_2$$

with $p \in \mathbf{x}_1(U)$ and $\varphi(\mathbf{x}_1(U_1)) \subset \mathbf{x}_2(U_2)$, the map

$$\mathbf{x}_2^{-1} \circ \varphi \circ \mathbf{x}_1 : U_1 \to U_2$$

is differentiable at $q = \mathbf{x}_1^{-1}(p)$

Differentiable mappings



Diffeomorphisms

We say that two shapes are **diffeomorphic** if there exists a differentiable map between them, with a differentiable inverse. Such a map is called a **diffeomorphism** between the two surfaces.

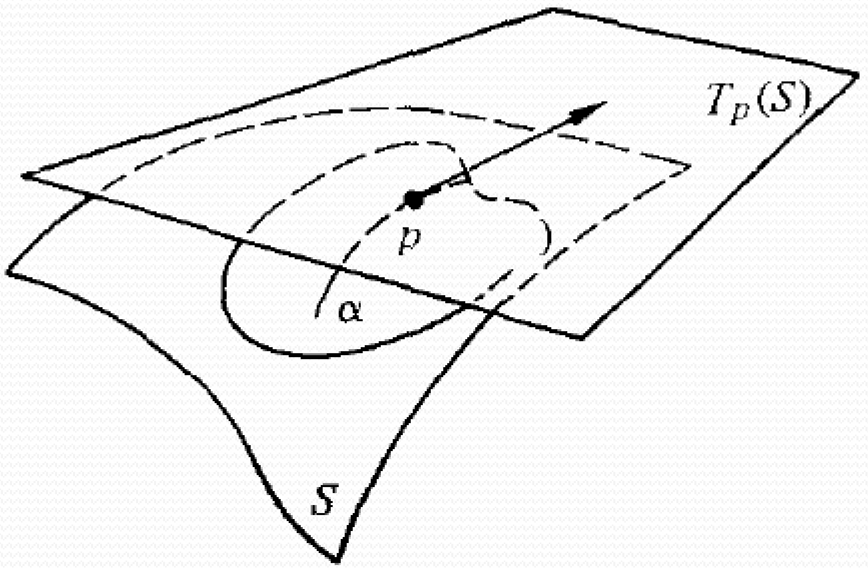
The notion of diffeomorphism plays the same role in the study of regular surfaces that the notion of isometry plays in the study of metric spaces.

From the point of view of differentiability, two diffeomorphic surfaces are indistinguishable.

Also note that every regular surface is locally diffeomorphic to a plane.

Tangent plane

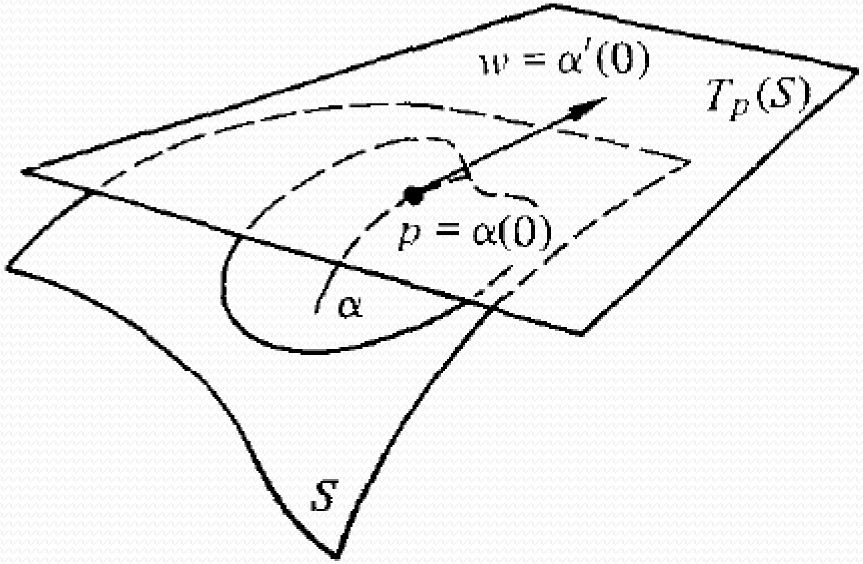
The set of tangent vectors to the parametrized curves of *S*, passing through *p*, constitutes the **tangent plane** at *p*. We will denote it by $T_p(S)$.



Tangent plane

Let us try to be more rigorous. First, note that given a tangent vector $w \in \mathbf{R}^3$ and a point $p_0 \in S \subset \mathbf{R}^3$, we can always find a differentiable curve $\alpha : (-\varepsilon, \varepsilon) \to S$ with $\alpha(0) = p_0$ and $\alpha'(0) = w$.

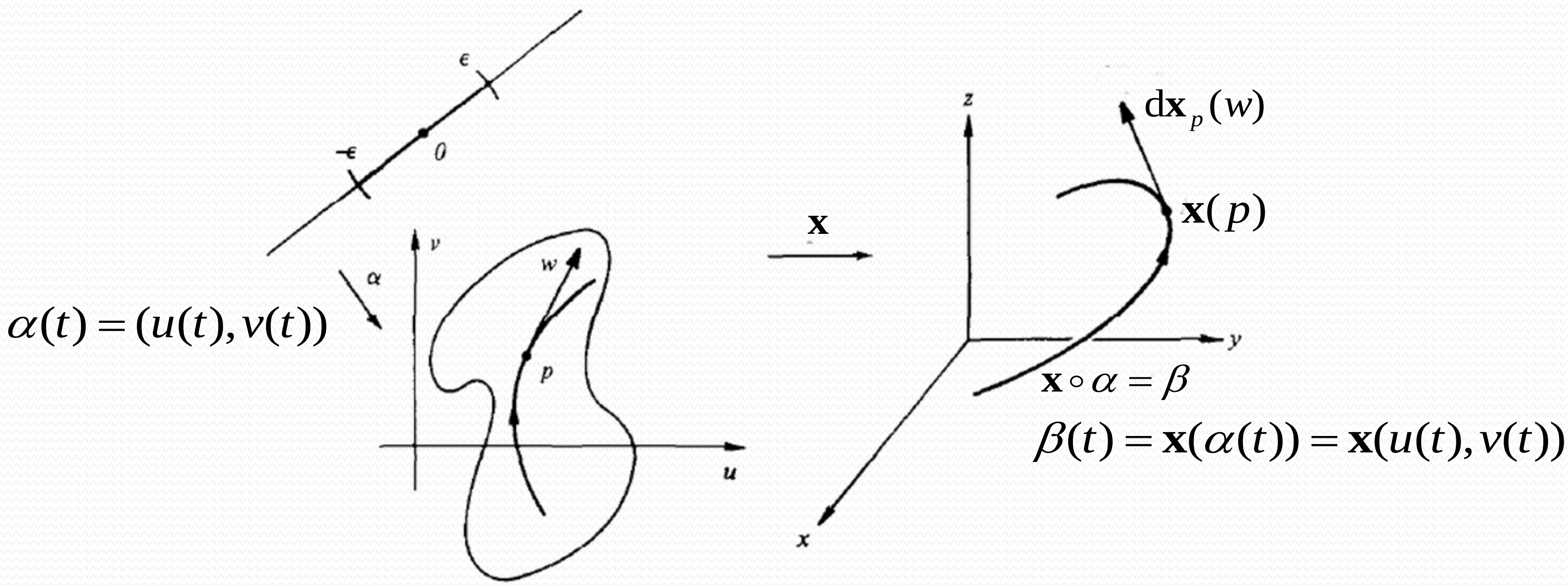
(simply write $\alpha(t) = p_0 + tw$)



Differential of a map

Now let $\mathbf{x}: U \subset \mathbf{R}^2 \to \mathbf{R}^3$ be a differentiable map, and let $\alpha : (-\varepsilon, \varepsilon) \to U$ be a differentiable curve on the parameter domain. Consider the differentiable curve $\beta = \mathbf{x} \circ \alpha : (-\varepsilon, \varepsilon) \to \mathbf{R}^3$. Then the **differential** of **x** at *p* is defined as:

$$\mathrm{d}\mathbf{x}_p(w) = \beta'(0)$$



Differential of a map

Now let $\mathbf{x}: U \subset \mathbf{R}^2 \rightarrow \mathbf{R}^3$ be a differentiable map, and let $\alpha: (-\varepsilon, \varepsilon) \rightarrow U$ be a differentiable curve on the parameter domain. Consider the differentiable curve $\beta = \mathbf{x} \circ \alpha: (-\varepsilon, \varepsilon) \rightarrow \mathbf{R}^3$. Then the **differential** of **x** at *p* is defined as:

$$\mathrm{d}\mathbf{x}_p(w) = \beta'(0)$$

• The differential is defined as $\mathrm{d}\mathbf{x}_p : \mathbf{R}^2 \rightarrow \mathbf{R}^3$, and is mapping tangent vectors to tangent vectors.

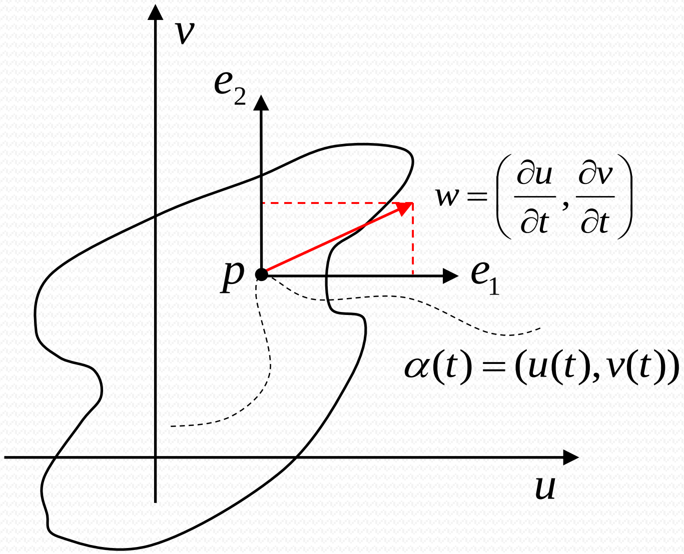
• The differential is a property of **x**, and as such it does *not* depend on the choice of the curve α .

• The differential is a linear map.

The latter two facts are made more evident in the next slide.

Differential of a map

Let (u,v) be coordinates in U and (x,y,z) be coordinates in $\mathbf{R}^3$. Then for the differentiable map $\mathbf{x}: U \subset \mathbf{R}^2 \to \mathbf{R}^3$, we have defined the differential as $d\mathbf{x}_p(w) = \beta'(0)$, where $\beta(t) = \mathbf{x}(\alpha(t)) = \mathbf{x}(u(t), v(t))$.
In order to differentiate β with respect to t, we apply the *chain rule* and obtain, in matrix form:



$$d\mathbf{x}_p(w) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} \end{pmatrix}$$

"Jacobian matrix" of $\mathbf{x}$ at p

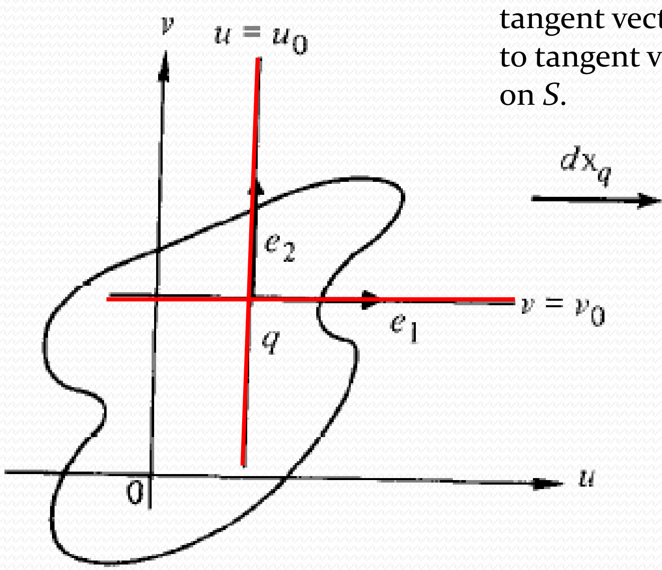
Notice that, indeed, the Jacobian matrix does not depend on the specific curve α that we introduced to define the differential.

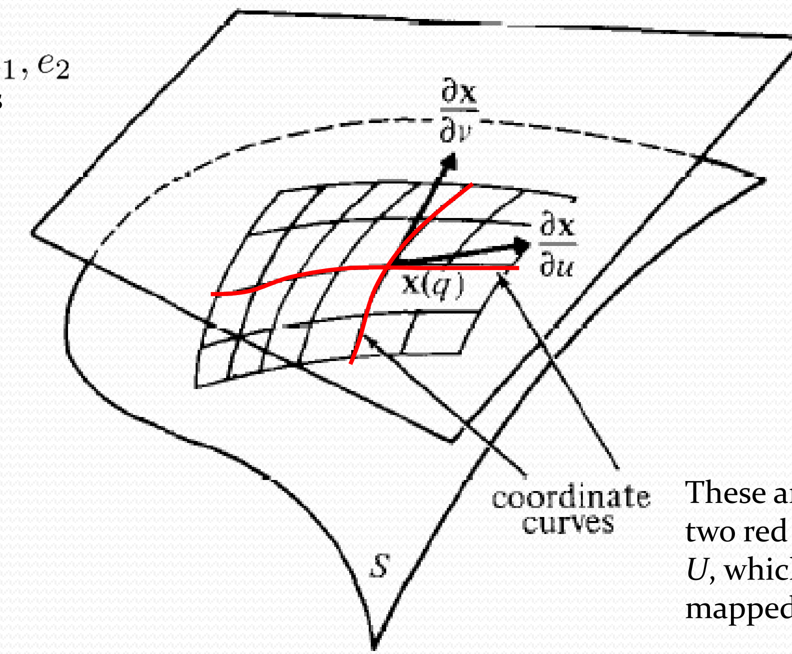
Differential of a map

Here $U \subset \mathbb{R}^2$ open and $\mathbf{x} : U \to S \subset \mathbb{R}^3$ regular if $\mathrm{d}\mathbf{x}_p : U \to \mathbb{R}^3$ has full rank.

$$\mathrm{d}\mathbf{x}_q \cdot e_1 = \begin{pmatrix} \vdots & \vdots \\ \frac{\partial \mathbf{x}}{\partial u} & \frac{\partial \mathbf{x}}{\partial v} \\ \vdots & \vdots \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\partial \mathbf{x}}{\partial u}$$

The differential is mapping the tangent vectors e_1, e_2 to tangent vectors on *S*.

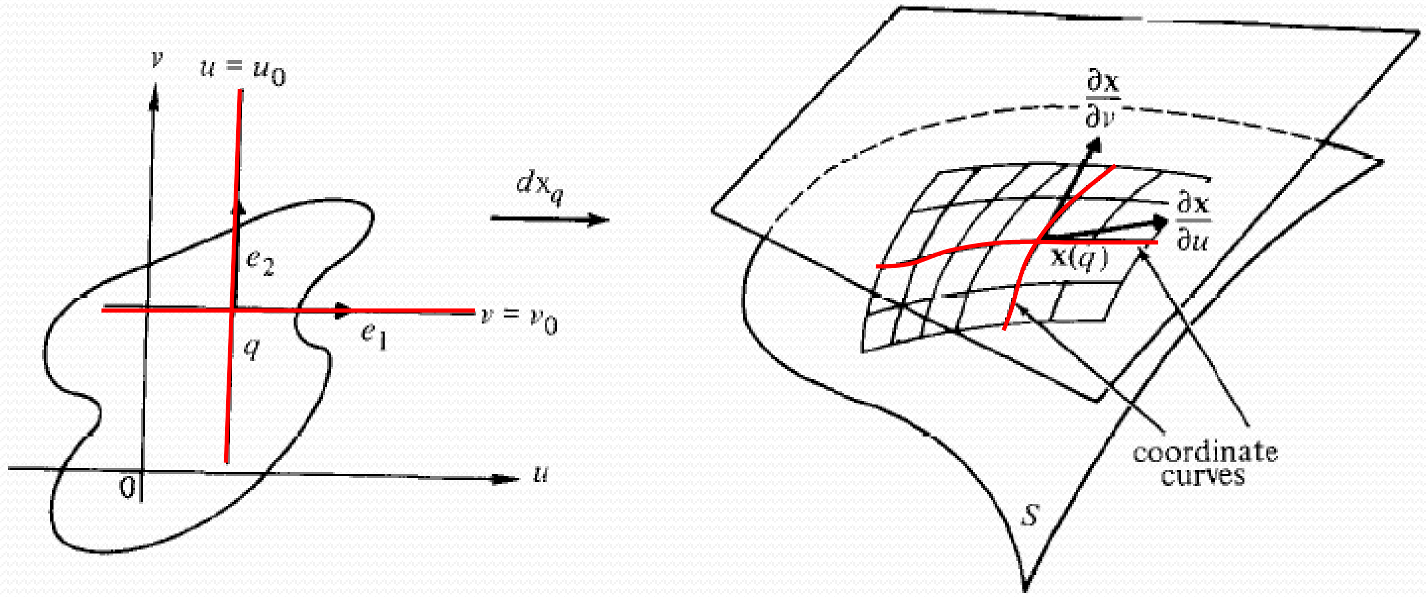




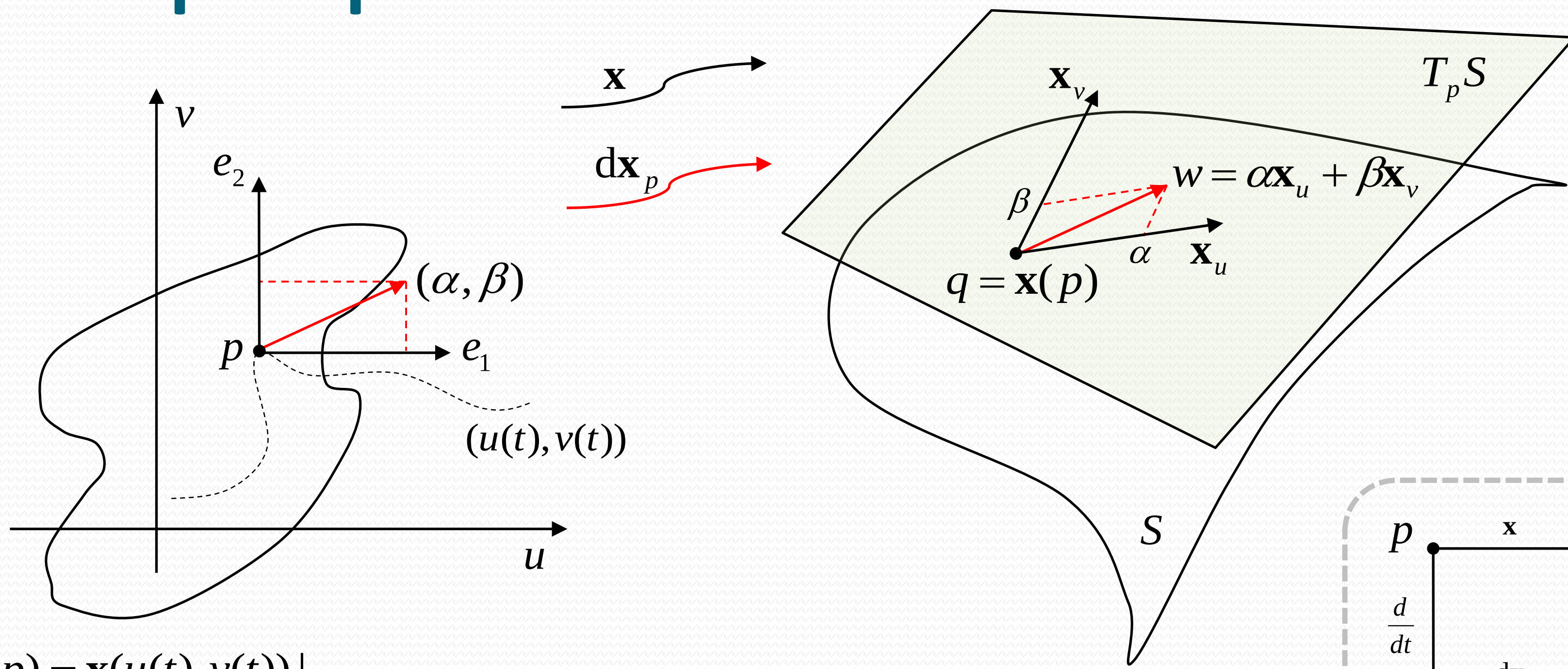
These are just the two red curves in *U*, which get mapped to *S* via **x**.

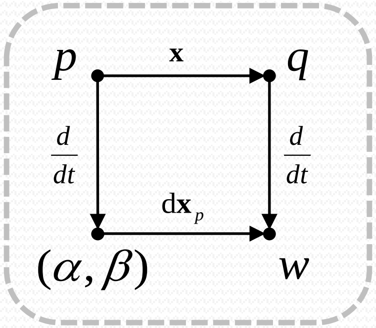
Tangent plane

We can now give a more rigorous definition for the tangent plane $T_p(S)$.
Let $\mathbf{x}: U \subset \mathbf{R}^2 \to S$ be a parametrization of a regular surface S and let $q \in U$.
The vector subspace of dimension 2, $d\mathbf{x}_q(\mathbf{R}^2) \subset \mathbf{R}^3$, coincides with the set of tangent vectors to S at $\mathbf{x}(q)$.



Wrap-up





$$q = \mathbf{x}(p) = \mathbf{x}(u(t), v(t))\,|_{t=0}$$

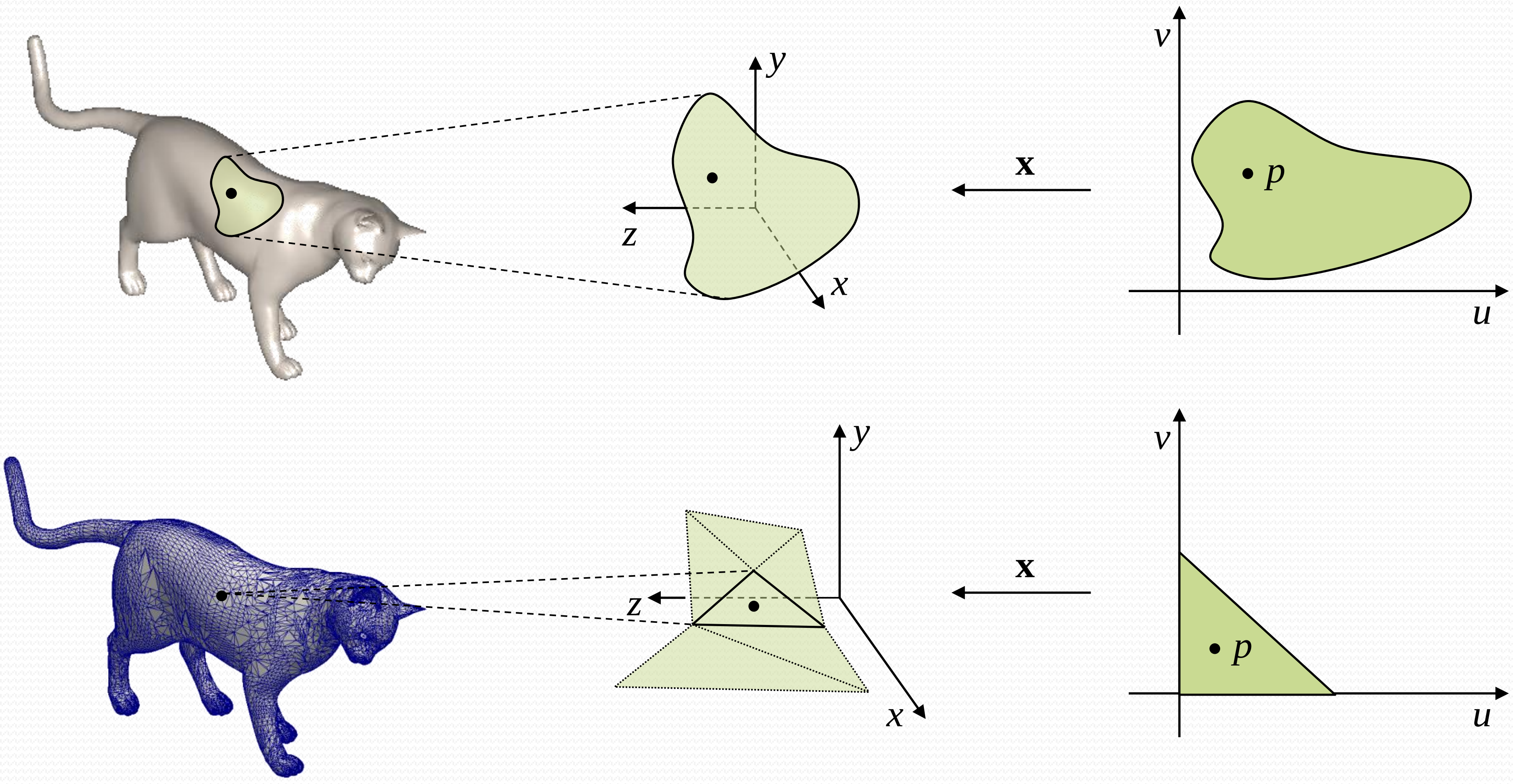
$$w = \frac{d}{dt}q\bigg|_{t=0} = \frac{d}{dt}\mathbf{x}(u(t), v(t))\,|_{t=0} = \frac{\partial \mathbf{x}}{\partial u}\frac{du}{dt} + \frac{\partial \mathbf{x}}{\partial v}\frac{dv}{dt} = \mathbf{x}_u u' + \mathbf{x}_v v' = \alpha\mathbf{x}_u + \beta\mathbf{x}_v$$

$$\mathrm{d}\mathbf{x}_p((\alpha, \beta)) = w$$

$$\mathrm{d}\mathbf{x}_p(e_1) = \mathbf{x}_u$$

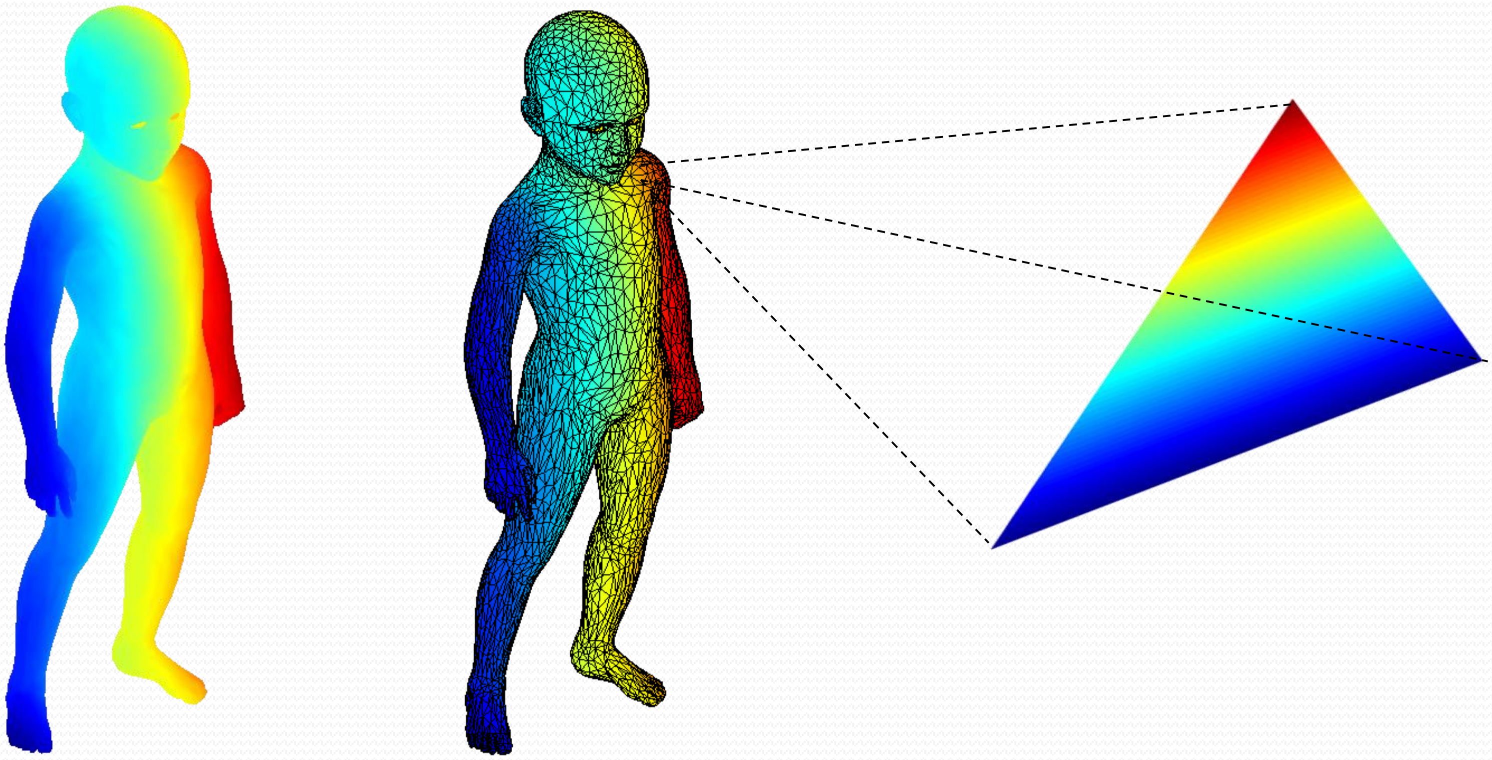
$$\mathrm{d}\mathbf{x}_p(e_2) = \mathbf{x}_v$$

Discretization



Piecewise-linear model

Given a triangle mesh, we will assume that scalar functions defined over it behave **piecewise-linearly** w.r.t. the triangulation. Note that this is a common assumption, but by no means the only possible model.

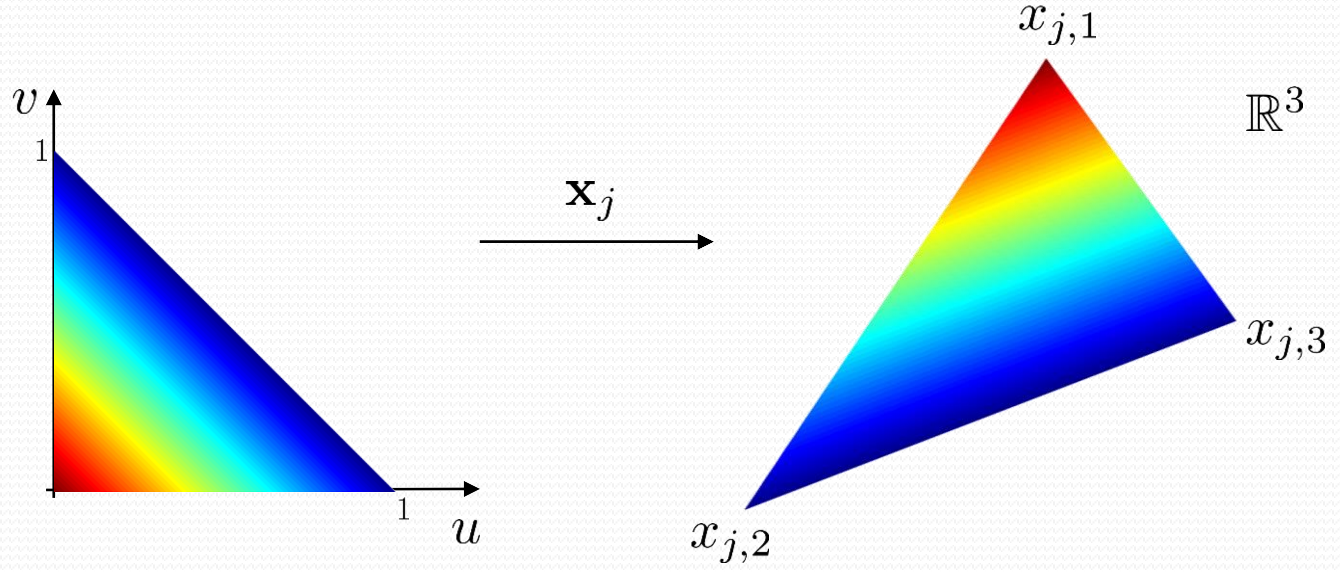


Mesh parametrization (triangles)

Let us consider a triangle mesh composed of *m* triangles. Our triangle-based parametrization is then described by the charts $\mathbf{x}_j : \mathbb{R}^2 \to \mathbb{R}^3$ for $j = 1, \ldots, m$

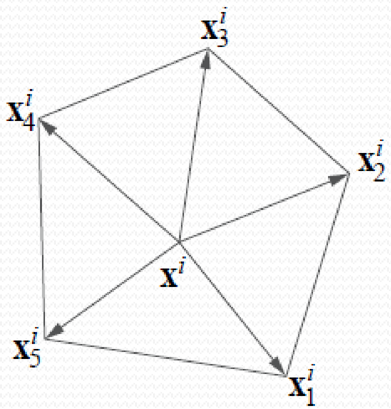
$$\mathbf{x}_j(u, v) : x_{j,1} + u(x_{j,2} - x_{j,1}) + v(x_{j,3} - x_{j,1})$$

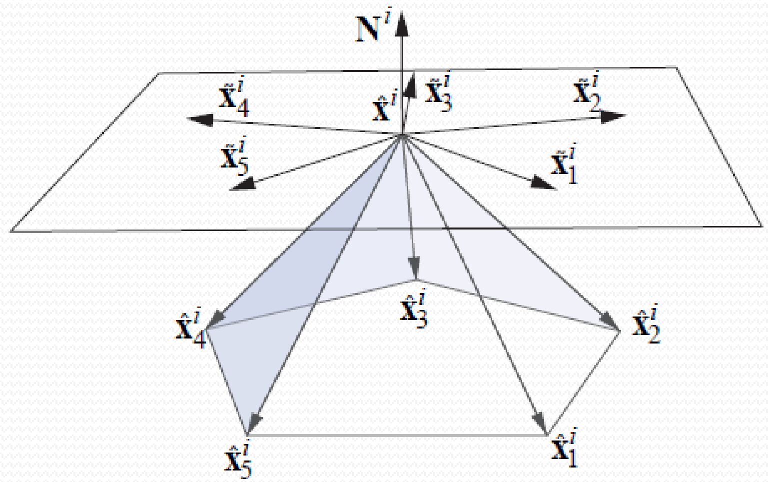
with $u \in [0, 1], v \in [0, 1 - \alpha]$.

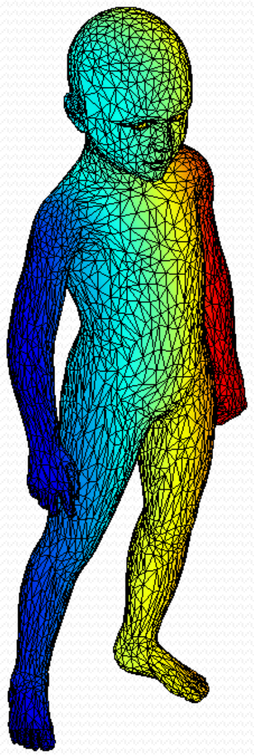


Mesh parametrization (1-rings)

Another possibility is to parametrize w.r.t. *patches* centered at each vertex. However, we will concentrate on the triangle-based parametrization throughout this course.







Suggested reading

- *Differential geometry of curves and surfaces*. Do Carmo – Chapters 2.1-2.4, Appendix 2.B
- *Differential Geometry: Curves – Surfaces – Manifolds*. W. Kühnel – Chapter 3A