Analysis of Three-Dimensional Shapes

(IN2238, TU München, Summer 2015)

The First Fundamental Form
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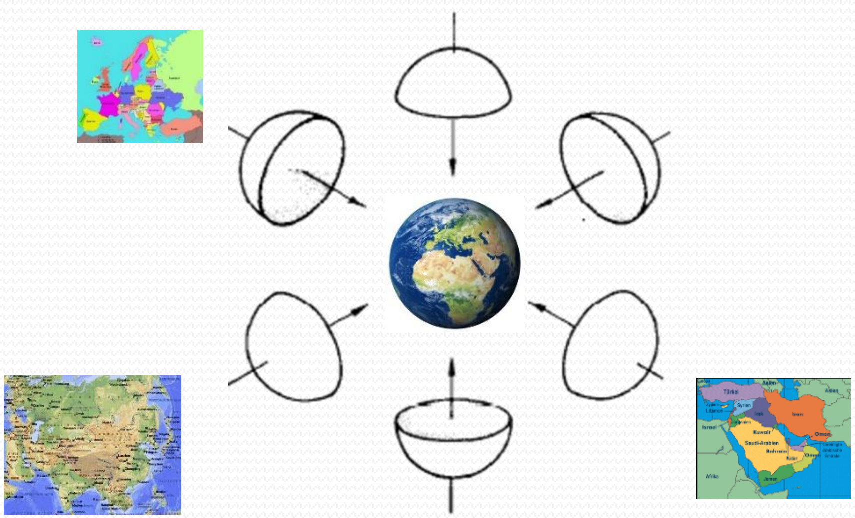
Wrap-up

Last time we have introduced the main notions of **differential geometry** that we will be using in this course.

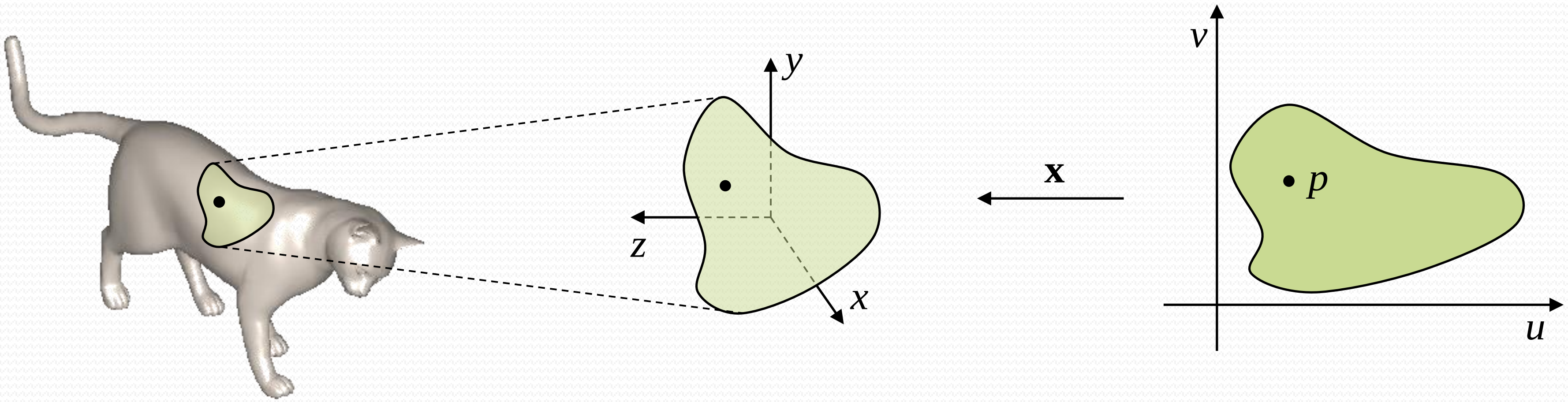
In particular, we showed how to model a 3D shape as a **regular surface**, that is, just a collection of deformed plane patches (called *surface elements*) glued together so as to form something smooth.



Wrap-up

The general idea of this approach is that we wish to analyze shapes according to a simple recipe:

- Consider each point of the shape as belonging to some **surface element**.
- Each surface element is the image of a known **diffeomorphism**, namely a **parametrization** function (or **chart**) $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3$
- Talking about the surface then corresponds to talking about **x**.

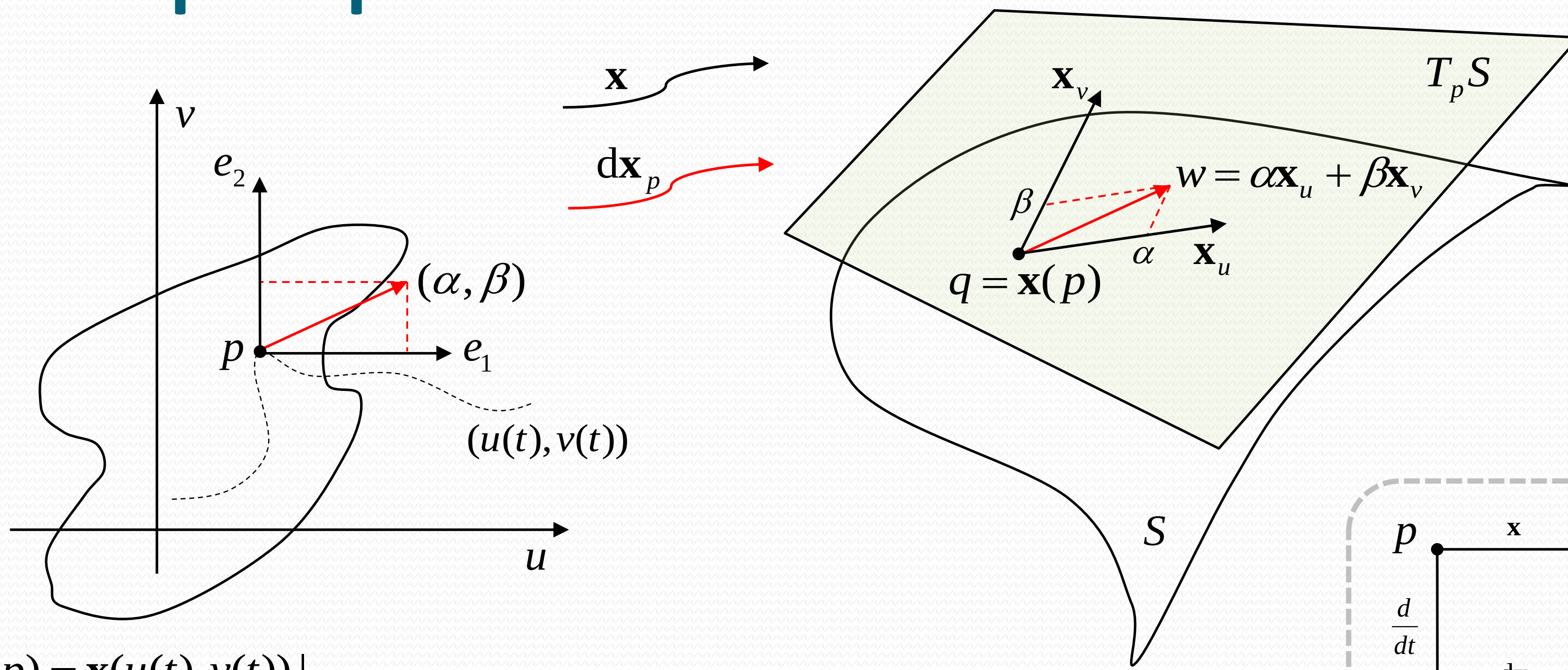


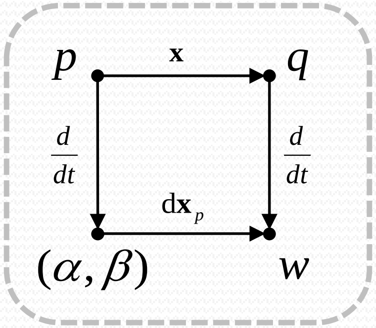
Wrap-up

In doing so, the are some **properties** that we naturally expect to be satisfied:

- The local properties of the surface should not depend on the specific choice of a parametrization $\mathbf{x}$.
- Since we want to speak about tangent planes, the parametrization should be *differentiable*.
- Since we know how to do calculus in $\mathbf{R}^n$, we would like to transfer this knowledge to the study of non-Euclidean domains (the surface).

Wrap-up





$$q = \mathbf{x}(p) = \mathbf{x}(u(t), v(t))\,|_{t=0}$$

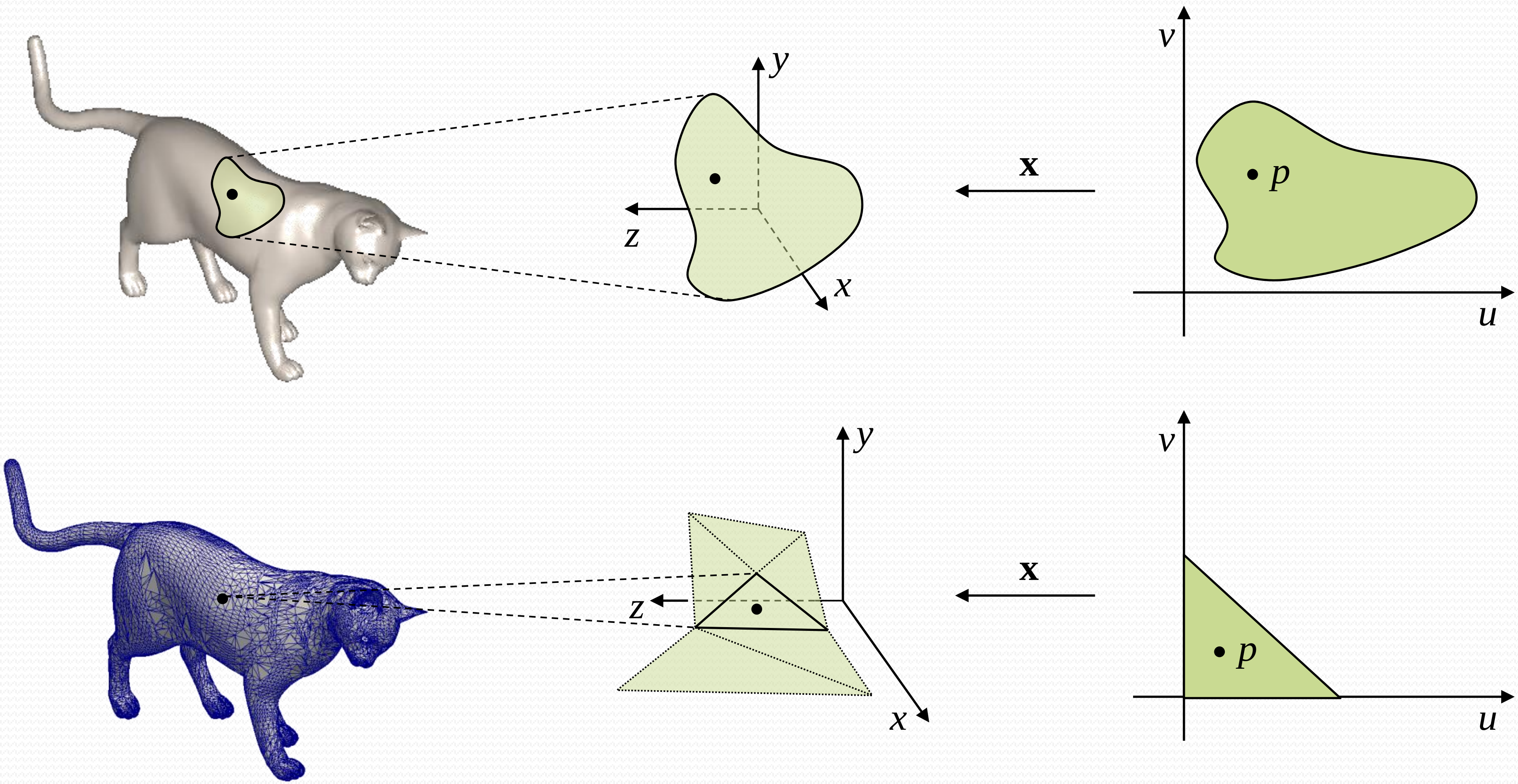
$$w = \frac{d}{dt} q \bigg|_{t=0} = \frac{d}{dt}\mathbf{x}(u(t), v(t))\,|_{t=0} = \frac{\partial \mathbf{x}}{\partial u}\frac{du}{dt} + \frac{\partial \mathbf{x}}{\partial v}\frac{dv}{dt} = \mathbf{x}_u u' + \mathbf{x}_v v' = \alpha\mathbf{x}_u + \beta\mathbf{x}_v$$

$$\mathrm{d}\mathbf{x}_p((\alpha, \beta)) = w$$

$$\mathrm{d}\mathbf{x}_p(e_1) = \mathbf{x}_u$$

$$\mathrm{d}\mathbf{x}_p(e_2) = \mathbf{x}_v$$

Wrap-up

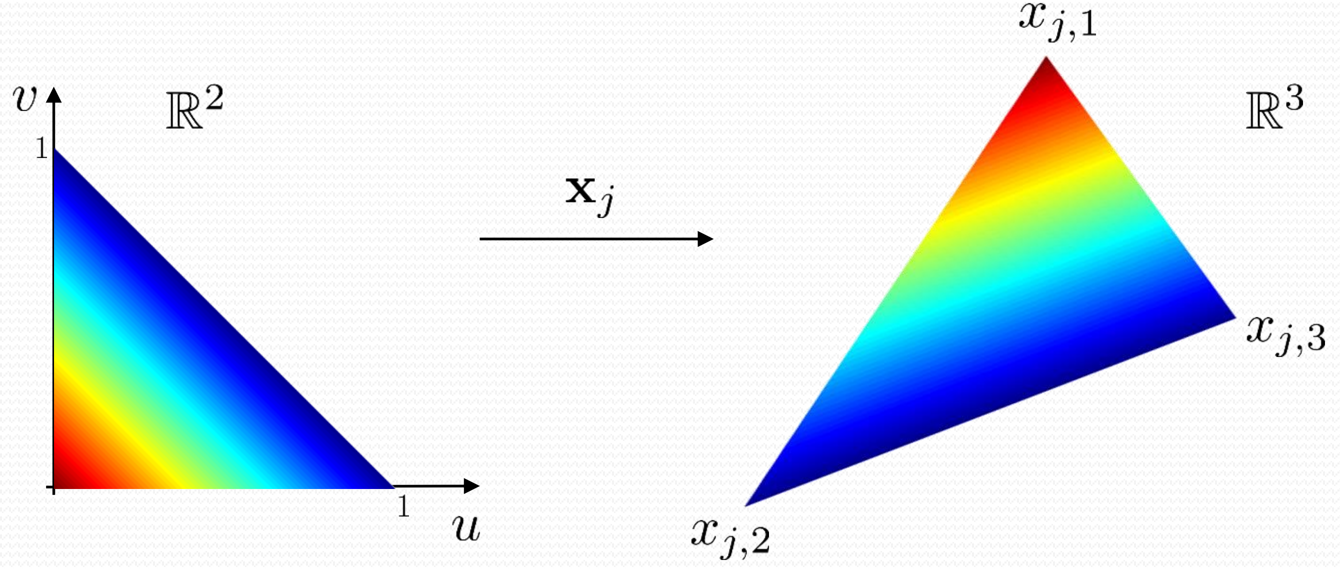


Wrap-up

Let us consider a triangle mesh composed of *m* triangles. Our triangle-based parametrization is then described by the charts $\mathbf{x}_j : \mathbb{R}^2 \to \mathbb{R}^3$ for $j = 1, \ldots, m$

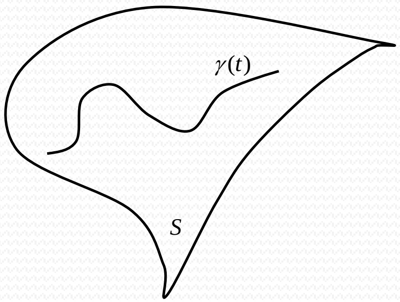
$$\mathbf{x}_j(u, v) = x_{j,1} + u(x_{j,2} - x_{j,1}) + v(x_{j,3} - x_{j,1})$$

with $u \in [0, 1], v \in [0, 1 - u]$.

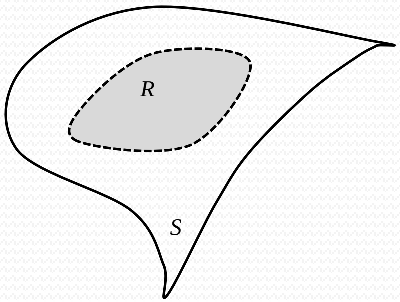


Measuring lengths and areas

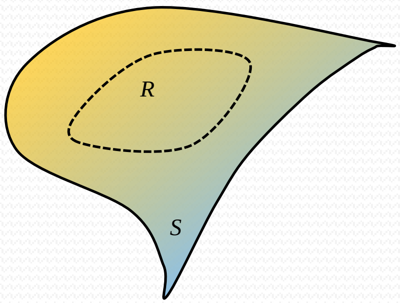
We are going to introduce a tool for measuring **metric properties** of a surface, such as *lengths*, *areas*, and *integrals* of scalar functions.



Length of a curve



Area of a region



Integral of a function

$f : S \to \mathbf{R}$

First fundamental form

The quadratic form $I_p : T_p(S) \to \mathbf{R}$ given by

$$I_p(w) = \langle w, w \rangle_p = \|w\|^2$$

is called the **first fundamental form** of the regular surface *S* at *p*.

We write $\langle \cdot,\cdot \rangle_p$ to remind ourselves that we are computing the inner product on the tangent plane at *p*.

The first fundamental form is, intuitively, the expression of how the surface *S* "inherits" the natural inner product of $\mathbf{R}^3$.

First fundamental form

Let us denote by $\{\mathbf{x}_u, \mathbf{x}_v\}$ the basis associated to a parametrization $\mathbf{x}(u,v)$ at p (thus, $\{\mathbf{x}_u, \mathbf{x}_v\}$ spans the tangent plane $T_p(S)$).

Any vector $w \in T_p(S)$ is the tangent vector to a curve $\alpha(t) = \mathbf{x}(u(t), v(t))$ which lies on the surface, with $t \in (-\varepsilon, \varepsilon)$ and $p = \alpha(0)$.

Then we can write:

chain rule

$$I_p(w) = I_p(\alpha'(0)) = \langle \alpha'(0), \alpha'(0) \rangle_p = \langle \mathbf{x}_u u' + \mathbf{x}_v v', \mathbf{x}_u u' + \mathbf{x}_v v' \rangle_p$$

$$= \langle \mathbf{x}_u, \mathbf{x}_u \rangle_p (u')^2 + 2\langle \mathbf{x}_u, \mathbf{x}_v \rangle_p u'v' + \langle \mathbf{x}_v, \mathbf{x}_v \rangle_p (v')^2$$

$$= E(u')^2 + 2Fu'v' + G(v')^2$$

First fundamental form

$$I_p(w) = E(u')^2 + 2Fu'v' + G(v')^2$$

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle_p$$
$$F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle_p$$
$$G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle_p$$

$$\Rightarrow \quad I_p(w) = \begin{pmatrix} u' & v' \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}$$

also called (Riemannian) **metric** tensor

E, *F*, and *G* are often called the "coefficients" or "components" of the first fundamental form. These coefficients play important roles in many intrinsic quantities of the surface.

By letting *p* run in the neighborhood defined by $\mathbf{x}(u,v)$ we obtain smooth functions $E(u,v)$, $F(u,v)$, $G(u,v)$. A manifold together with this smooth inner product is called a **Riemannian manifold**.

First fundamental form

$$I_p(w) = E(u')^2 + 2Fu'v' + G(v')^2$$

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle_p$$

$$F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle_p$$

$$G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle_p$$

$$\Rightarrow \quad I_p(w) = \begin{pmatrix} u' & v' \end{pmatrix} \underbrace{\begin{pmatrix} E & F \\ F & G \end{pmatrix}}_{} \begin{pmatrix} u' \\ v' \end{pmatrix}$$

this is often called just g

We have seen that the differential map associated to **x** is represented by the Jacobian matrix:

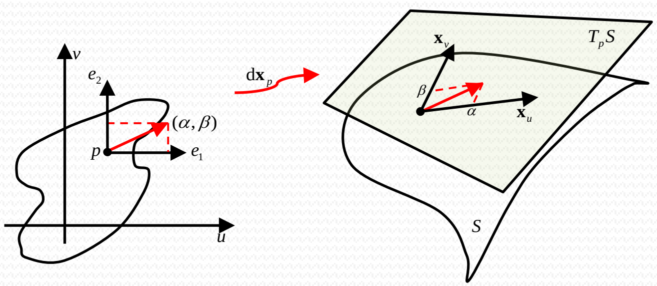
$$\mathrm{D}\mathbf{x} = \begin{pmatrix} \vdots & \vdots \\ \mathbf{x}_u & \mathbf{x}_v \\ \vdots & \vdots \end{pmatrix}$$

Then, it is easy to see that:

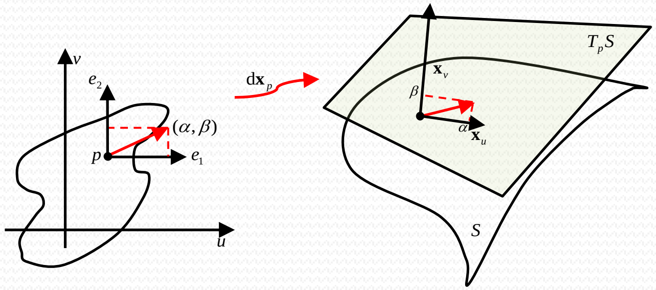
$$g = \mathrm{D}\mathbf{x}^{\mathrm{T}}\mathrm{D}\mathbf{x} = \begin{pmatrix} \langle \mathbf{x}_u, \mathbf{x}_u \rangle & \langle \mathbf{x}_u, \mathbf{x}_v \rangle \\ \langle \mathbf{x}_v, \mathbf{x}_u \rangle & \langle \mathbf{x}_v, \mathbf{x}_v \rangle \end{pmatrix}$$

Parametrizations

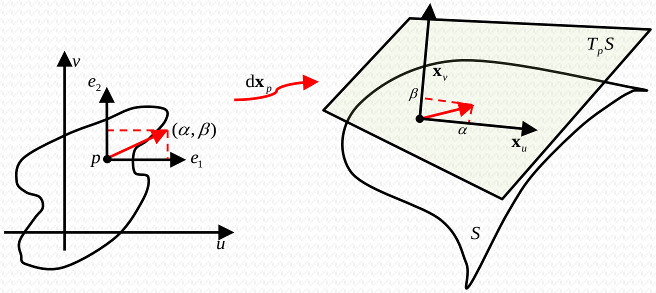
$$I_p(w) = \begin{pmatrix} u' & v' \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} \qquad E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle_p \quad F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle_p \quad G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle_p$$



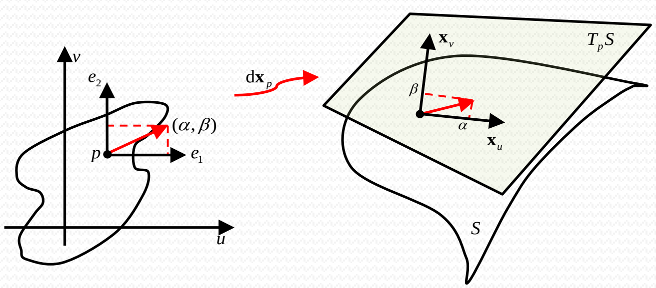
generic parametrization



orthogonal parametrization $F \equiv 0$



conformal parametrization $F \equiv 0\,, E = G$



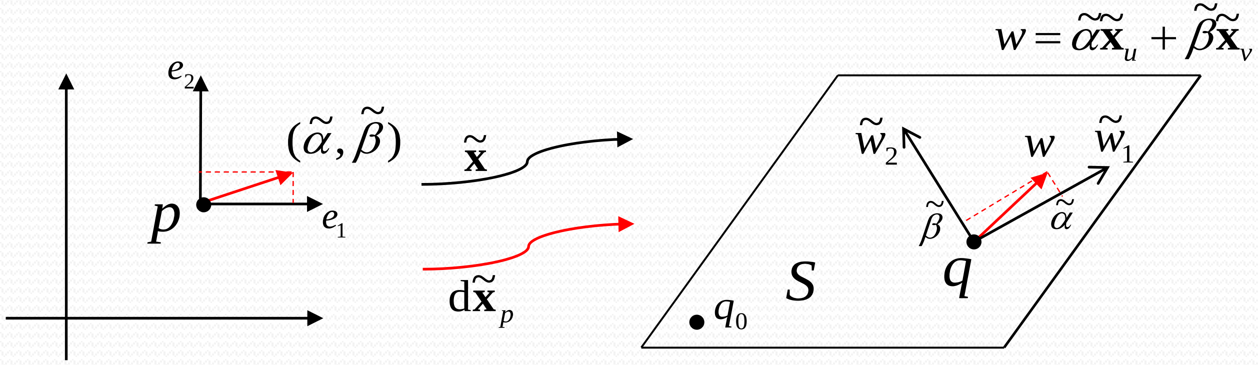
isometric parametrization $F \equiv 0\,, E = G = 1$

The confusing example

Consider a plane $S \subset \mathbf{R}^3$ passing through q_0 and containing the *orthonormal* vectors $\tilde{w}_1$ and $\tilde{w}_2$.

$$\tilde{\mathbf{x}}(u,v) = q_0 + u\tilde{w}_1 + v\tilde{w}_2 \quad \Rightarrow \quad \begin{aligned} \tilde{\mathbf{x}}_u &= \tilde{w}_1 \\ \tilde{\mathbf{x}}_v &= \tilde{w}_2 \end{aligned}$$

We want to compute the first fundamental form for an arbitrary point *q* in *S*.



$$\tilde{g} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Euclidean metric tensor

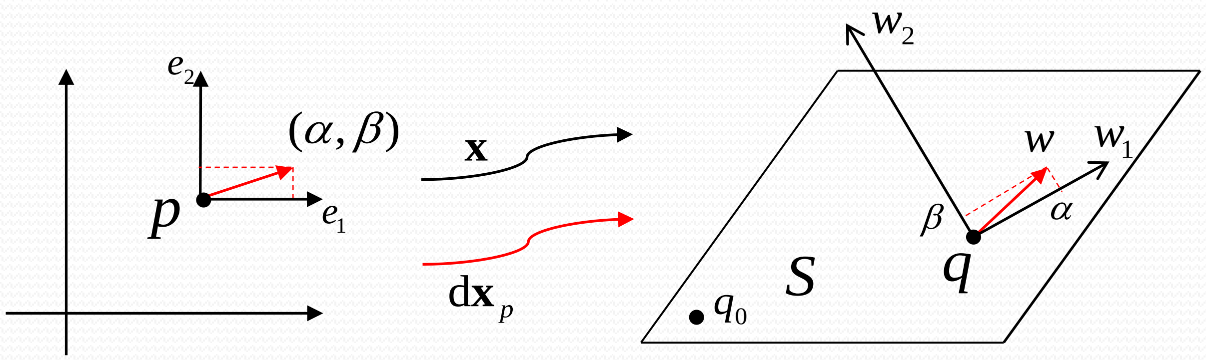
Thus, the first fundamental form of *w* at *p* is $I_p((\tilde{\alpha}, \tilde{\beta})) = \begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} = \tilde{\alpha}^2 + \tilde{\beta}^2$

Example 2 (plane)

Consider the previous example, but this time let $\|w_1\| = 1$ and $\|w_2\| = 2$.
We are changing the parametrization $\mathbf{x}$, but still we expect that the lengths of vectors in $T_p(S)$ do *not* change (as they are a property of the <u>surface</u>).

Say, for example, that we take the same (p,w) from the previous example.

As before, we have $\mathbf{x}_u = w_1,\ \mathbf{x}_v = w_2$, and then $g = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$.



previous example: $w = \tilde{\alpha}\tilde{\mathbf{x}}_u + \tilde{\beta}\tilde{\mathbf{x}}_v$

this example: $w = \alpha\mathbf{x}_u + \beta\mathbf{x}_v$

The two bases, and thus the coefficients for w are <u>different</u> in the two examples.

$$\alpha\mathbf{x}_u + \beta\mathbf{x}_v \overset{!}{=} \tilde{\alpha}\tilde{\mathbf{x}}_u + \tilde{\beta}\tilde{\mathbf{x}}_v$$

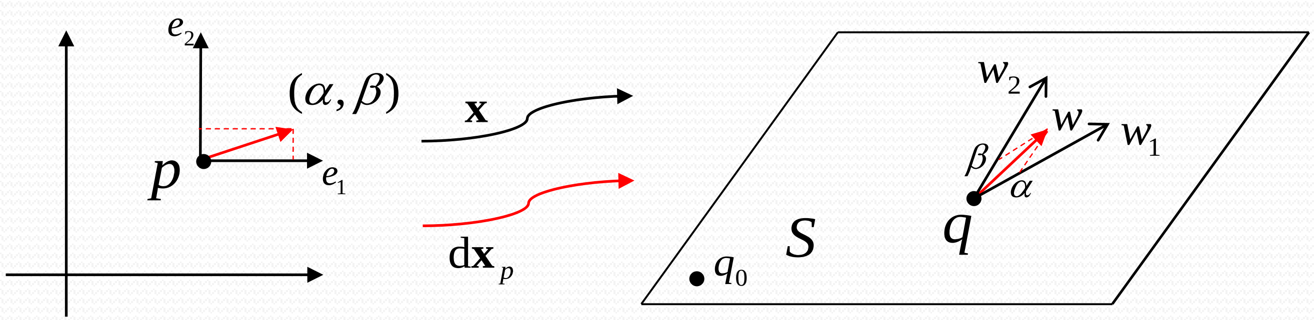
We can now compute $I_p((\alpha,\beta)) = \begin{pmatrix} \alpha & \beta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \tilde{\alpha} & \frac{\tilde{\beta}}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} \tilde{\alpha} \\ \frac{\tilde{\beta}}{2} \end{pmatrix} = \tilde{\alpha}^2 + \tilde{\beta}^2$

Example 3 (plane)

Let's make it more interesting and let $\|w_1\| = 1, \ \|w_2\| = 1,$ and $\langle w_1, w_2 \rangle = \frac{1}{\sqrt{2}}$.
Again, we expect that the length of *w* does *not* change.

Once again, we have $\mathbf{x}_u = w_1, \ \mathbf{x}_v = w_2$, and now $g = \begin{pmatrix} 1 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1 \end{pmatrix}$.

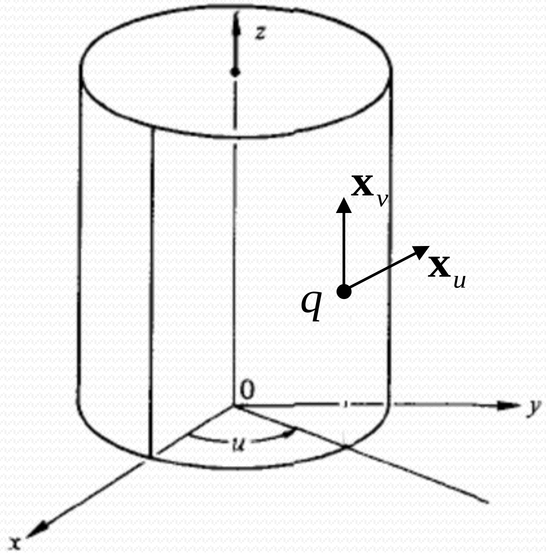
Even though the coefficients in *g* are different, again we expect the first fundamental form to be the same as before.



$$\alpha w_1 + \beta w_2 \overset{!}{=} \tilde{\alpha} \tilde{w}_1 + \tilde{\beta} \tilde{w}_2$$
$$\Downarrow$$
$$w_1 = \tilde{w}_1, \quad w_2 = \frac{1}{\sqrt{2}} (\tilde{w}_1 + \tilde{w}_2)$$
$$\alpha = \tilde{\alpha} - \tilde{\beta}, \quad \beta = \sqrt{2} \tilde{\beta}$$

So we get $I_p((\alpha, \beta)) = \begin{pmatrix} \alpha & \beta \end{pmatrix} \begin{pmatrix} 1 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \tilde{\alpha} - \tilde{\beta} & \sqrt{2}\tilde{\beta} \end{pmatrix} \begin{pmatrix} 1 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} \tilde{\alpha} - \tilde{\beta} \\ \sqrt{2}\tilde{\beta} \end{pmatrix} = \tilde{\alpha}^2 + \tilde{\beta}^2$

Example 4 (cylinder)



$$\mathbf{x}(u,v) = (\cos u, \sin u, v)$$

$$U = \{(u,v) \in \mathbf{R}^2;\ 0 < u < 2\pi,\ -\infty < v < \infty\}$$

$$\mathbf{x}_u = (-\sin u, \cos u, 0),\ \ \mathbf{x}_v = (0,0,1)$$

$$E = \sin^2 u + \cos^2 u = 1$$

$$F = 0$$

$$G = 1$$

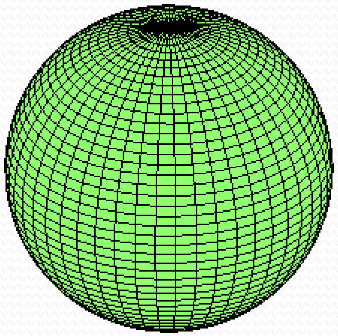
$$\Rightarrow\ g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We notice that the plane and the cylinder behave locally in the same way, since their first fundamental forms are equal.

In other words, plane and cylinder are *locally isometric*. We will discuss about local isometries more in detail at a later time.

Example 5a (sphere)

$$\mathbf{x} : (0, 2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}^3 \qquad \mathbf{x}(u,v) = \begin{pmatrix} \cos(u)\cos(v) \\ \sin(u)\cos(v) \\ \sin(v) \end{pmatrix}$$



$$\mathrm{D}\mathbf{x} = \begin{pmatrix} \vdots & \vdots \\ \mathbf{x}_u & \mathbf{x}_v \\ \vdots & \vdots \end{pmatrix} = \begin{pmatrix} -\sin(u)\cos(v) & -\cos(u)\sin(v) \\ \cos(u)\cos(v) & -\sin(u)\sin(v) \\ 0 & \cos(v) \end{pmatrix}$$

$$g = \mathrm{D}\mathbf{x}^{\mathrm{T}}\mathrm{D}\mathbf{x} = \begin{pmatrix} \cos^2(v) & 0 \\ 0 & 1 \end{pmatrix}$$

From this example it becomes evident that the coefficients *E*, *F*, *G* are indeed differentiable functions *E*(*u*,*v*), *F*(*u*,*v*), *G*(*u*,*v*).

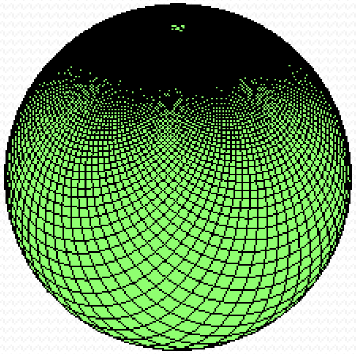
Thus, if $w = \alpha \mathbf{x}_u + \beta \mathbf{x}_v$ is the tangent vector to the sphere at point **x**(*u*,*v*), then its squared length is given by $|w|^2 = I(w) = \alpha^2 \cos^2(v) + \beta^2$.

Example 5b (sphere)

$\mathbf{y} : \mathbb{R}^2 \to \mathbb{R}^3$

$$\mathbf{y}(\tilde{u}, \tilde{v}) = \frac{1}{\tilde{u}^2 + \tilde{v}^2 + 1} \begin{pmatrix} 2\tilde{u} \\ 2\tilde{v} \\ \tilde{u}^2 + \tilde{v}^2 - 1 \end{pmatrix}$$

$$\mathrm{D}\mathbf{y}(\tilde{u}, \tilde{v}) = \frac{2}{(\tilde{u}^2 + \tilde{v}^2 + 1)^2} \begin{pmatrix} \tilde{v}^2 - \tilde{u}^2 + 1 & 2\tilde{u}\tilde{v} \\ 2\tilde{u}\tilde{v} & \tilde{u}^2 - \tilde{v}^2 + 1 \\ 2\tilde{u} & 2\tilde{v} \end{pmatrix}$$



$$g = \mathrm{D}\mathbf{y}^{\mathrm{T}} \mathrm{D}\mathbf{y}$$

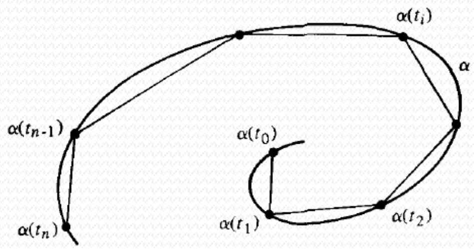
The result is probably going to look not very nice.

In general, from a computational point of view it is much more convenient to plug in the values for $\tilde{u}, \tilde{v}$ directly in $\mathrm{D}\mathbf{y}(\tilde{u}, \tilde{v})$, and only then compute g.

Length of a curve

By knowing the first fundamental form, we can treat metric questions on a regular surface without further references to the ambient space.

arc-length of a curve $\alpha:(0,T)\to S$ $\quad s(t)=\int_0^t \|\alpha'(x)\|dx=\int_0^t \sqrt{I(\alpha'(x))}dx$



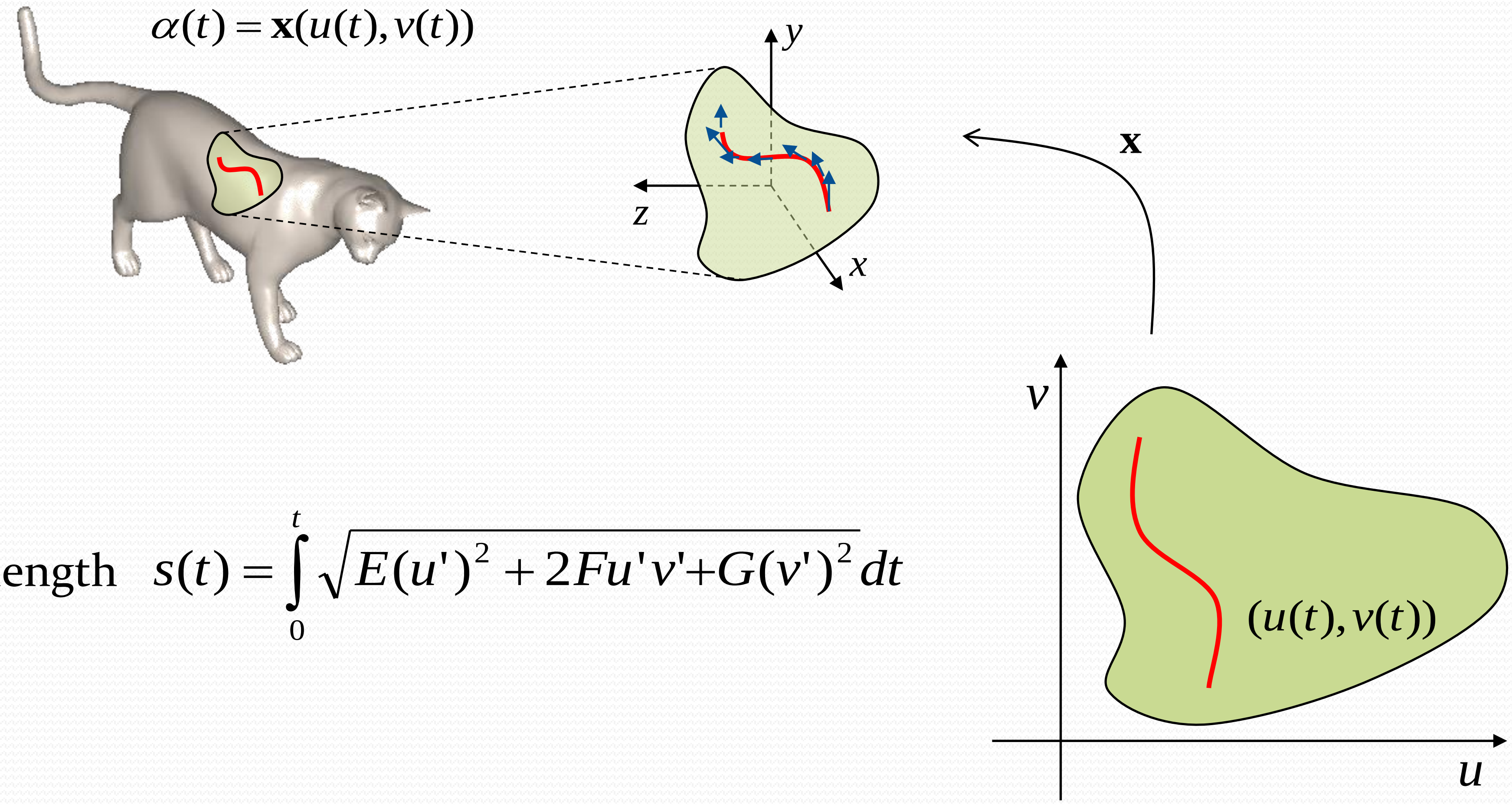
Remember that E, F, G are actually functions of (u,v), so in general they are changing along the curve.

Thus, if $\alpha(t)=\mathbf{x}(u(t),v(t))$ is contained in a surface element parametrized by $\mathbf{x}(u,v)$, we can compute the length as:

$$s(t)=\int_0^t \sqrt{E(u')^2+2Fu'v'+G(v')^2}\,dt$$

Length of a curve

$$\alpha(t) = \mathbf{x}(u(t), v(t))$$



length $\quad s(t) = \int_0^t \sqrt{E(u')^2 + 2Fu'v' + G(v')^2}\, dt$

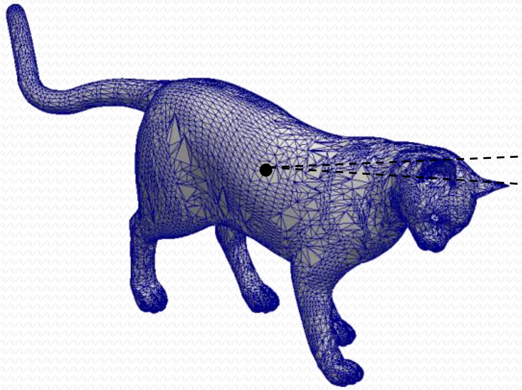
Length of a curve

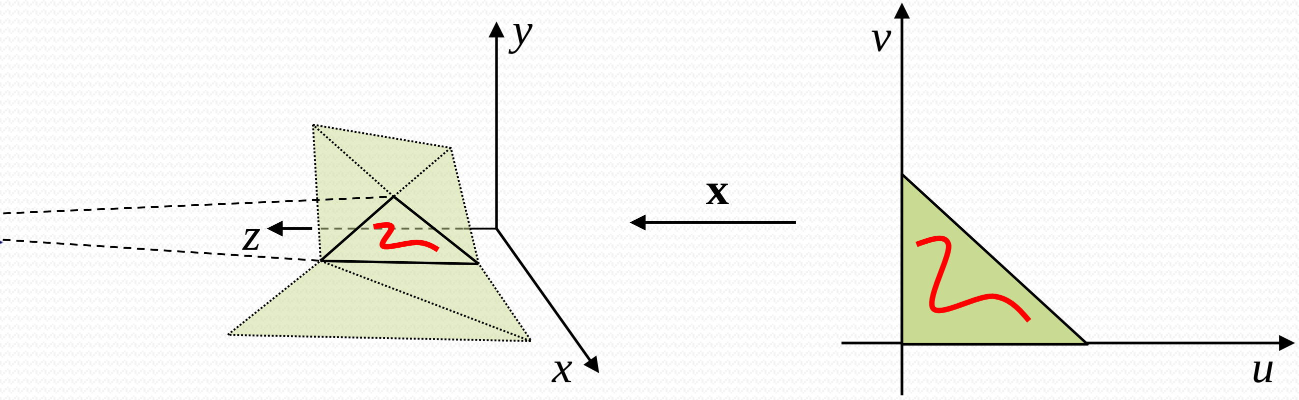
Pitfall: Notice that we are talking about length of curves **within** surface elements, thus in general this construction cannot be used to compute distances among *any two* given surface points.

In fact, we wrote:

Thus, if $\alpha(t) = \mathbf{x}(u(t), v(t))$ is contained in a surface element parametrized by $\mathbf{x}(u,v)$, we can compute the length as:

$$s(t) = \int_0^t \sqrt{E(u')^2 + 2Fu'v' + G(v')^2}\, dt$$





Arc length element

Length $s(t) = \int_0^t \|\alpha'(x)\| dx$

The first fundamental theorem of calculus gives us: $\frac{ds}{dt} = \|\alpha'(t)\|$

⇩

$ds = \|\alpha'(t)\| dt$

A more compact notation for the length of a curve can then be defined as:

$$\text{length}(\alpha) = \int_\alpha ds$$

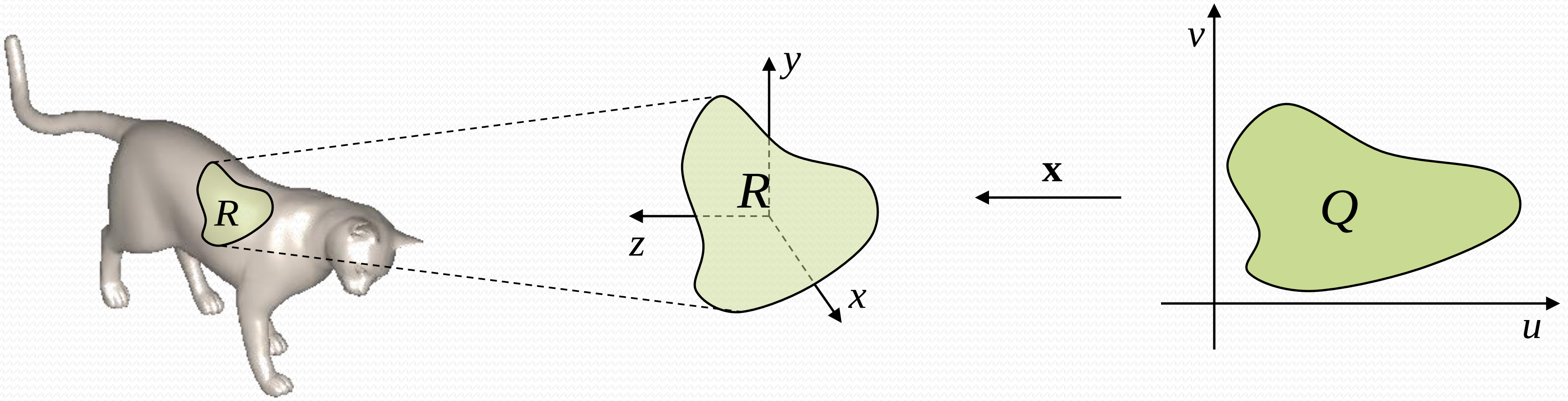
In terms of the metric tensor, the **arc length element** *ds* is given by:

$$ds = \sqrt{Edu^2 + 2Fdudv + Gdv^2}\, dt$$

Area of a region

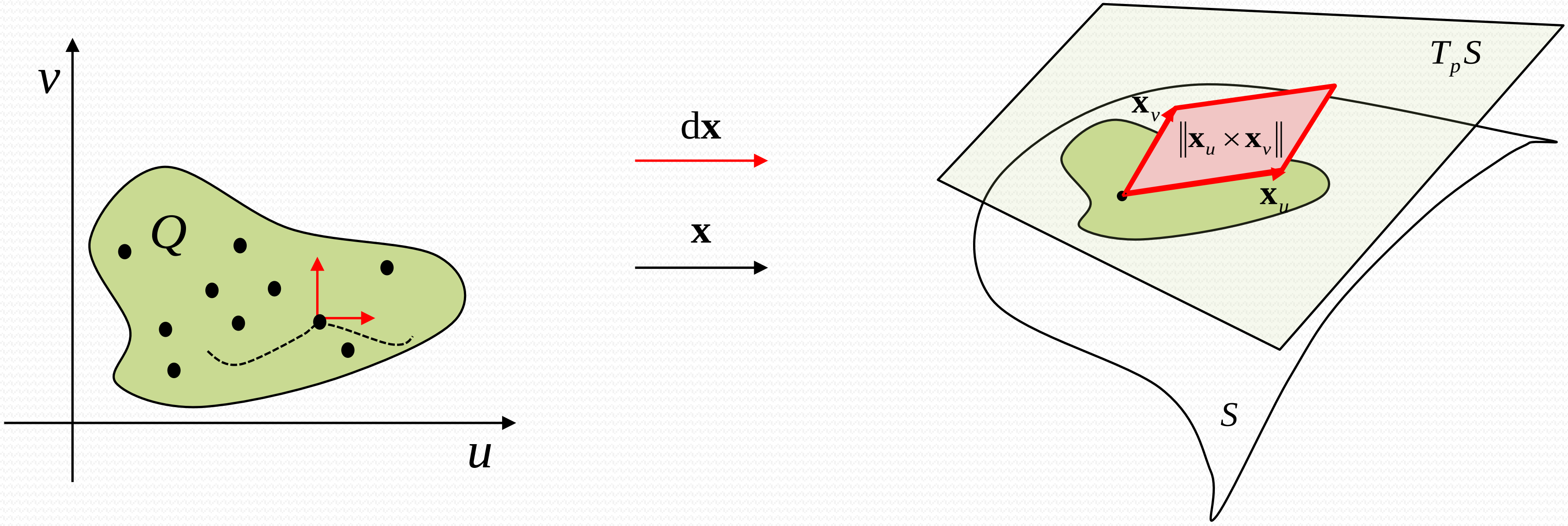
The first fundamental form can be employed to compute the area of a bounded region R of a regular surface S. If $R \subset S$ is contained in the image of the parametrization $\mathbf{x}: U \subset \mathbf{R}^2 \to S$, the **area** of R is **defined** by

$$A(R) = \iint_Q \left\| \mathbf{x}_u \times \mathbf{x}_v \right\| du dv, \qquad Q = \mathbf{x}^{-1}(R)$$



Area of a region

$$A(R) = \iint_Q \|\mathbf{x}_u \times \mathbf{x}_v\| du dv, \qquad Q = \mathbf{x}^{-1}(R)$$



The area of a region on the surface is defined as the sum of the areas of parallelograms tangent to that surface region.

Area of a region

$$A(R) = \iint_Q \|\mathbf{x}_u \times \mathbf{x}_v\| dudv, \qquad Q = \mathbf{x}^{-1}(R) \qquad g = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

Observe that:

$$\|\mathbf{x}_u \times \mathbf{x}_v\|^2 = \|\mathbf{x}_u\|^2 \|\mathbf{x}_v\|^2 \sin^2 \omega = \|\mathbf{x}_u\|^2 \|\mathbf{x}_v\|^2 (1 - \cos^2 \omega) = \|\mathbf{x}_u\|^2 \|\mathbf{x}_v\|^2 - \langle \mathbf{x}_u, \mathbf{x}_v \rangle^2$$

We can rewrite:

$$\|\mathbf{x}_u \times \mathbf{x}_v\| = \sqrt{\|\mathbf{x}_u\|^2 \|\mathbf{x}_v\|^2 - \langle \mathbf{x}_u, \mathbf{x}_v \rangle^2} = \sqrt{EG - F^2} = \sqrt{\det g}$$

And so we come to the more compact writing:

$$A(R) = \iint_Q \sqrt{\det g}\, dudv$$

Area element

Similarly to the arc length case, we can define the **area element** *da* as:

$$da = \sqrt{\det g}\, du dv$$

And then we can use the following notation:

$$A(R) = \int_R da$$

The area element is also called **(Riemannian) volume form**. In the case of 2-dimensional manifolds (our case), volume corresponds to area.

Wrap-up

We have obtained two alternative expressions for measuring lengths and areas: one is defined in **parameter space**, the other is defined directly on the **surface**.

Parameter space

$$\text{length}(\alpha) = \int_0^T \|\alpha'(t)\| dt = \int_0^T \sqrt{Edu^2 + 2Fdudv + Gdv^2}\, dt$$

Surface

$$\text{length}(\alpha) = \int_\alpha ds \qquad ds = \sqrt{Edu^2 + 2Fdudv + Gdv^2}\, dt$$

Parameter space

$$A(R) = \iint_Q \|\mathbf{x}_u \times \mathbf{x}_v\| dudv, \qquad Q = \mathbf{x}^{-1}(R)$$

Surface

$$A(R) = \int_R da \qquad da = \sqrt{\det g}\, dudv$$

Integral of a function

We can follow a similar approach to compute the integral of a function defined over the surface, $f : S \to \mathbf{R}$

Let us use our newly introduced notation:

$$\int_R f(x)dx$$

Analogously to the previous slide, we get to the **definition**:

$$\int_R f(x)dx = \iint_Q f(\mathbf{x}(u,v))\sqrt{\det g}\,dudv, \qquad Q = \mathbf{x}^{-1}(R)$$

$$\int_{\phi(U)} f(\mathbf{v})\,d\mathbf{v} = \int_U f(\phi(\mathbf{u}))\,|\det(\mathrm{D}\phi)(\mathbf{u})|\,d\mathbf{u}.$$

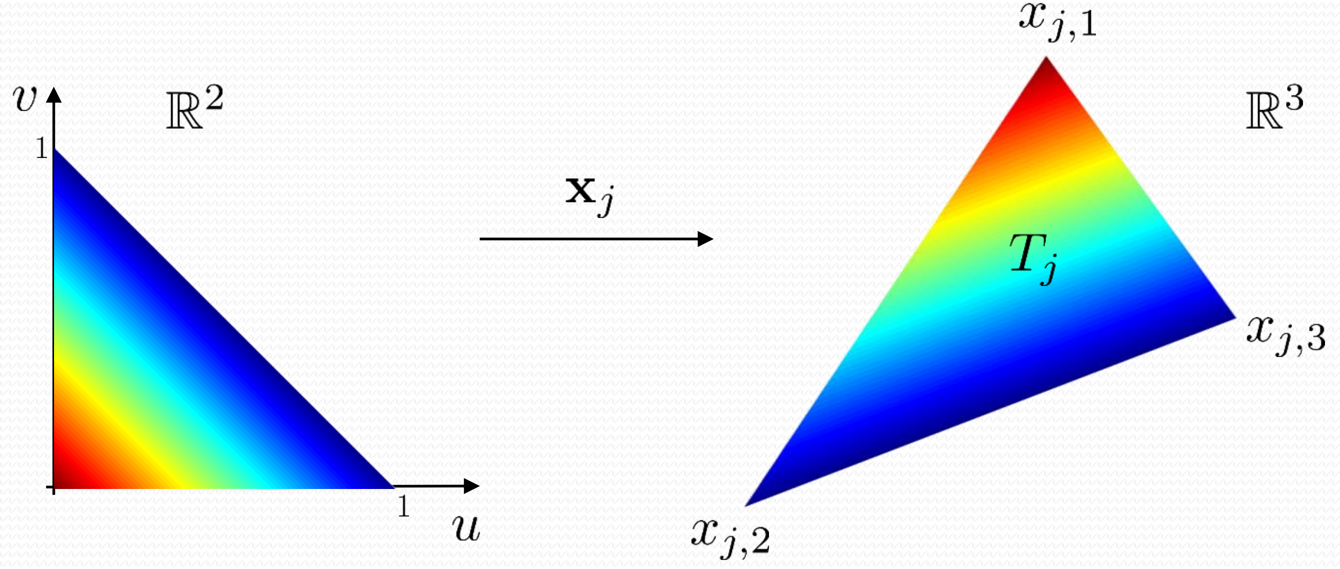
Generalizes the substitution rule in classical multivariate calculus

Discretization: chart

Let us consider a triangle mesh composed of *m* triangles. Our triangle-based parametrization is then described by the charts $\mathbf{x}_j : \mathbb{R}^2 \to \mathbb{R}^3$ for $j = 1, \ldots, m$

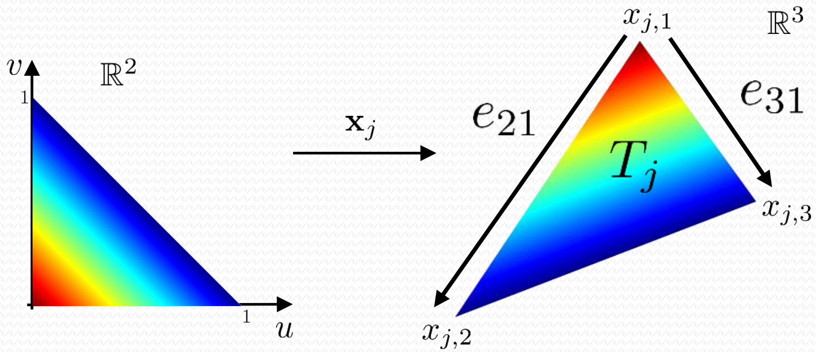
$$\mathbf{x}_j(u, v) = x_{j,1} + u(x_{j,2} - x_{j,1}) + v(x_{j,3} - x_{j,1})$$

with $u \in [0, 1], v \in [0, 1 - u]$.



Discretization: metric tensor

$$\mathbf{x}_j(u,v) = x_{j,1} + u(x_{j,2} - x_{j,1}) + v(x_{j,3} - x_{j,1})$$



We simply have:

$$\mathbf{x}_u = x_{j,2} - x_{j,1} = e_{21}$$

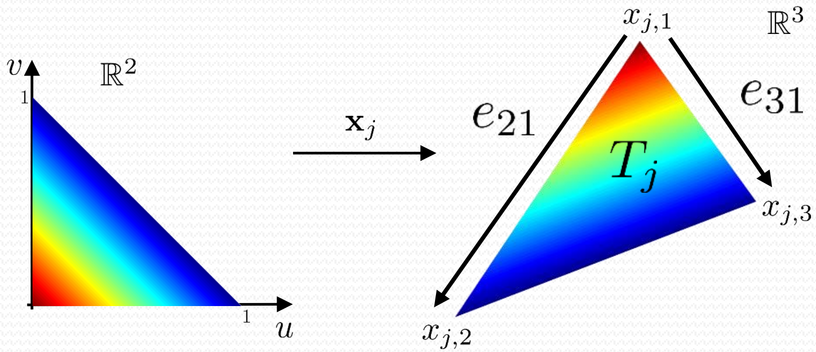
$$\mathbf{x}_v = x_{j,3} - x_{j,1} = e_{31}$$

The coefficients for the metric tensor / first fundamental form are thus given by:

$$g_j = \begin{pmatrix} E_j & F_j \\ F_j & G_j \end{pmatrix} = \begin{pmatrix} \langle \mathbf{x}_u, \mathbf{x}_u \rangle & \langle \mathbf{x}_u, \mathbf{x}_v \rangle \\ \langle \mathbf{x}_v, \mathbf{x}_u \rangle & \langle \mathbf{x}_v, \mathbf{x}_v \rangle \end{pmatrix} = \begin{pmatrix} \|e_{21}\|^2 & \langle e_{21}, e_{31} \rangle \\ \langle e_{21}, e_{31} \rangle & \|e_{31}\|^2 \end{pmatrix}$$

Discretization: area element

$$\mathbf{x}_j(u,v) = x_{j,1} + u(x_{j,2} - x_{j,1}) + v(x_{j,3} - x_{j,1})$$



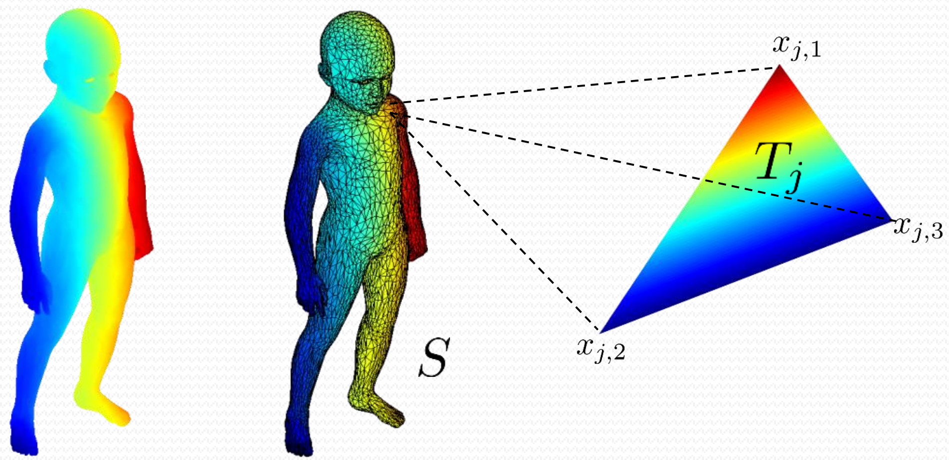
$$g_j = \begin{pmatrix} \|e_{21}\|^2 & \langle e_{21}, e_{31} \rangle \\ \langle e_{21}, e_{31} \rangle & \|e_{31}\|^2 \end{pmatrix}$$

Let us compute the area of the triangle by applying our definition of area of a region:

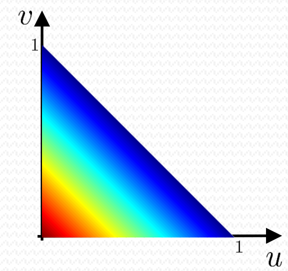
$$\int_{T_j} da = \int_0^1 \int_0^{1-u} \sqrt{\det g_j} du dv = 2A(T_j) \int_0^1 \int_0^{1-u} du dv = 2A(T_j)\frac{1}{2} = A(T_j)$$

Discretization: integral

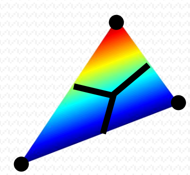
$$\mathbf{x}_j(u,v) = x_{j,1} + u(x_{j,2} - x_{j,1}) + v(x_{j,3} - x_{j,1})$$



$f : S \to \mathbb{R}$ behaves **linearly** within each triangle and it is uniquely determined by its values at the vertices of the triangle.

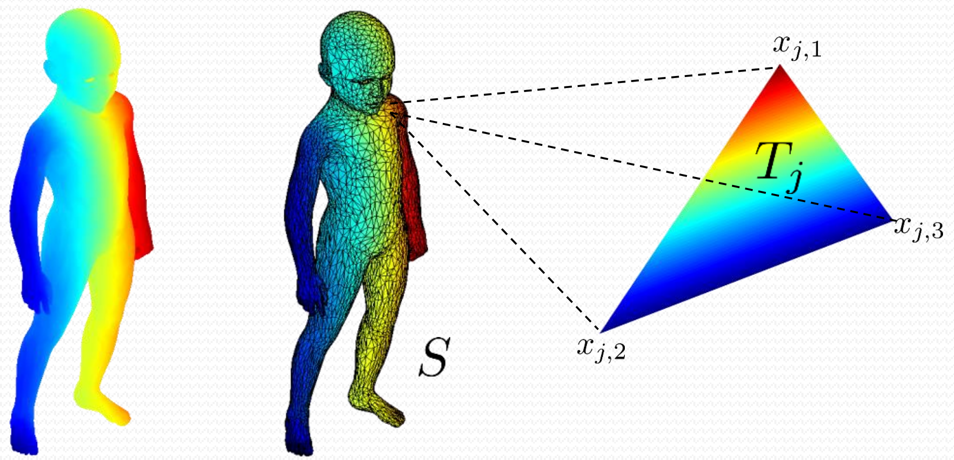


$$\begin{aligned}
\int_{T_j} f \; da &= \int_0^1 \int_0^{1-u} f(\mathbf{x}(u,v))\sqrt{\det g_j}\, du dv \\
&= \int_0^1 \int_0^{1-u} f(x_{j,1})(1-u-v) + f(x_{j,2})u + f(x_{j,3})v\sqrt{\det g_j}\, du dv \\
&= \tfrac{1}{6}(f(x_{j,1}) + f(x_{j,2}) + f(x_{j,3}))2A(T_j) \\
&= \tfrac{1}{3}(f(x_{j,1}) + f(x_{j,2}) + f(x_{j,3}))A(T_j)
\end{aligned}$$



Discretization: integral

$$\mathbf{x}_j(u,v) = x_{j,1} + u(x_{j,2} - x_{j,1}) + v(x_{j,3} - x_{j,1})$$



$f : S \to \mathbb{R}$ behaves **linearly** within each triangle and it is uniquely determined by its values at the vertices of the triangle.

The integral of *f* over a region $R \subseteq S$ is just the sum of the integrals over each triangle T_j .

$$\int_R f \; da = \sum_{j=1}^{|R|} \int_{T_j} f \; da$$

Suggested reading

- *Differential geometry of curves and surfaces*. Do Carmo – Chapters 2.5, Appendix 2.B
- *Differential Geometry: Curves – Surfaces – Manifolds*. W. Kühnel – Chapter 3A