Analysis of Three-Dimensional Shapes (IN2238, TU München, Summer 2015)

Isometries
(05.05.2015)

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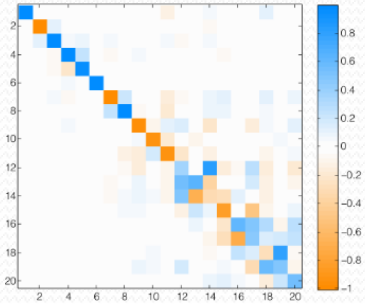
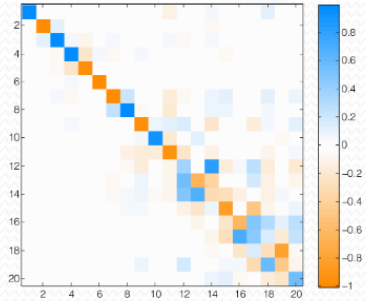
Room 02.09.058, Informatik IX

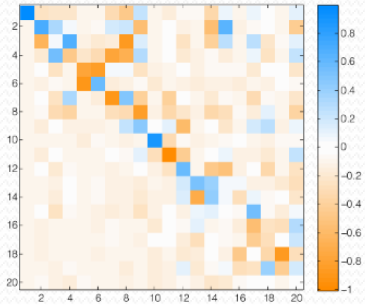
Seminar

"Functional Maps"
Simon Urbainczyk

Thursday, May 07th
14:00 Room 02.09.023







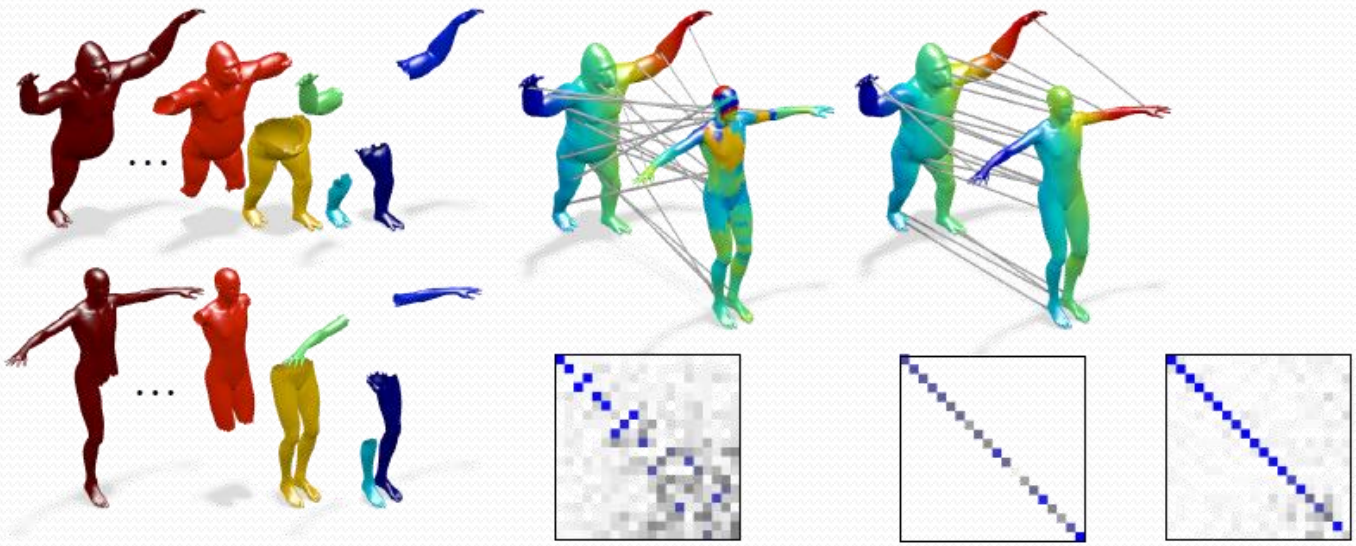
Seminar

"Coupled Quasi-Harmonic Bases"

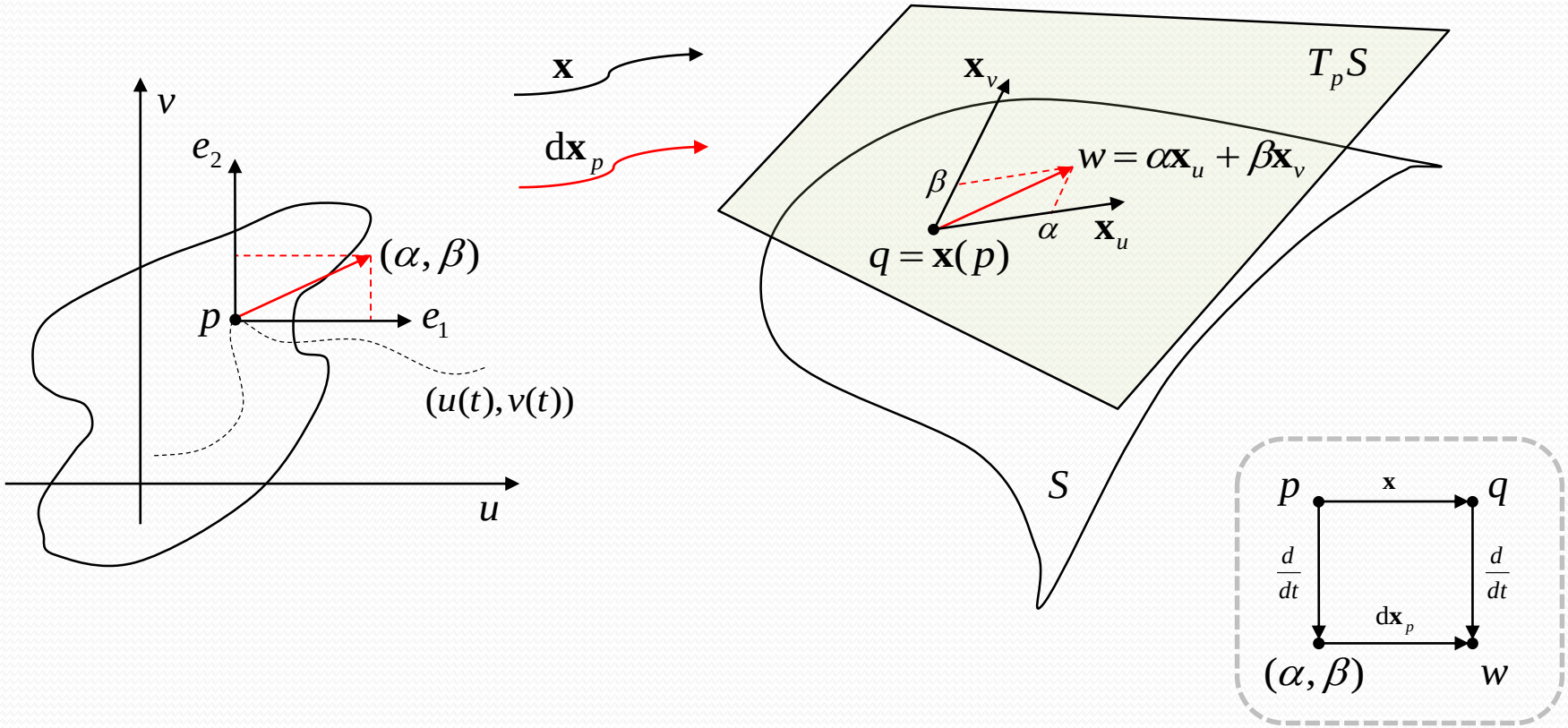
Alexander Binsmaier

Thursday, May 07th

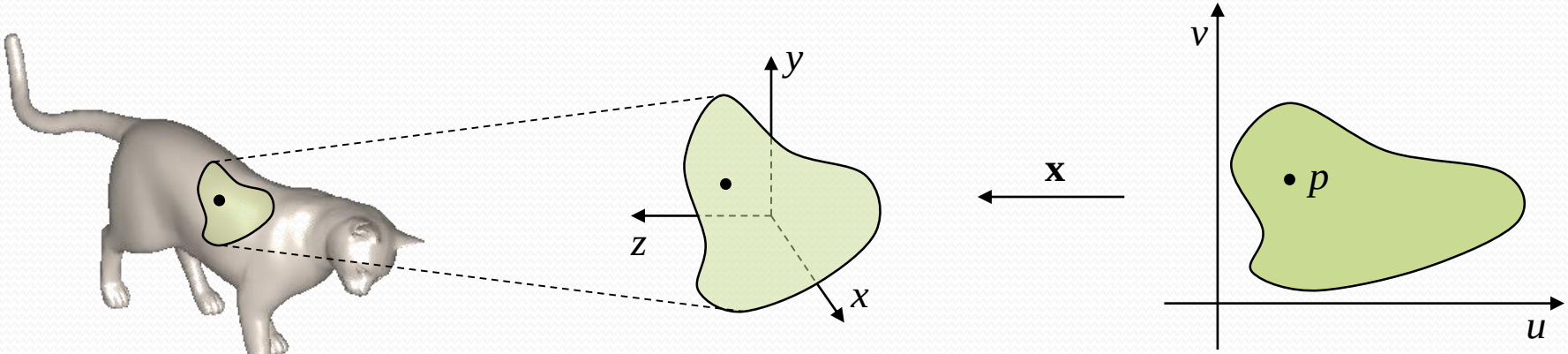
14:00 Room 02.09.023

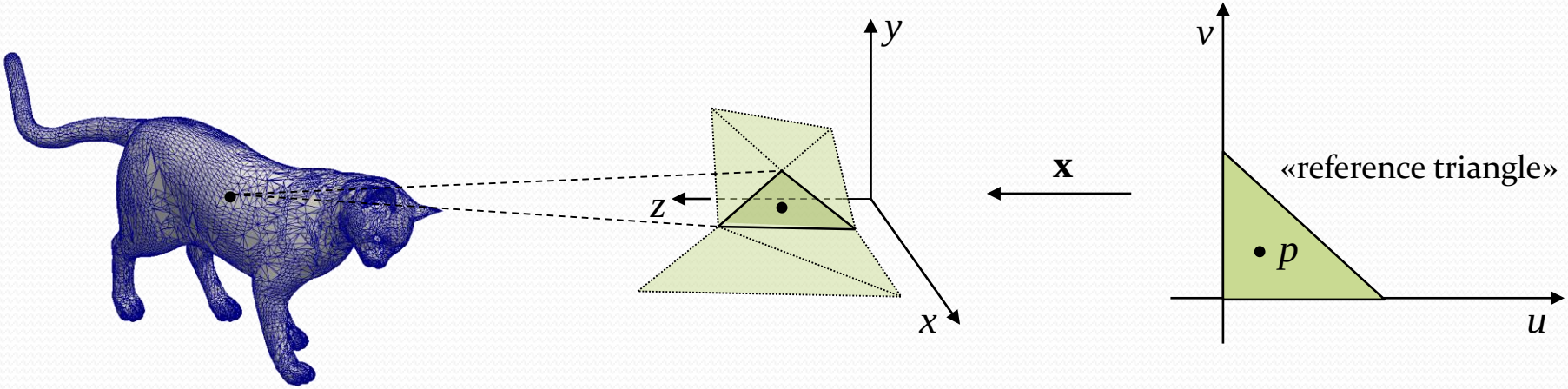


Wrap-up


v
e_2
(α, β)
p
e_1
$(u(t), v(t))$
u
$\mathbf{x}$
$\mathrm{d}\mathbf{x}_p$
$\mathbf{x}_v$
T_pS
$w = \alpha\mathbf{x}_u + \beta\mathbf{x}_v$
β
α
$\mathbf{x}_u$
$q = \mathbf{x}(p)$
S
p
$\mathbf{x}$
q
$\frac{d}{dt}$
$\frac{d}{dt}$
$\mathrm{d}\mathbf{x}_p$
(α, β)
w

Wrap-up





Wrap-up

We introduced the notion of **first fundamental form**, or **metric tensor** for our smooth surfaces.

$$I_p : T_p(S) \to \mathbf{R} \qquad\qquad I_p(w) = \langle w, w \rangle_p = \|w\|^2$$

This quantity can be defined in terms of certain coefficients, and expressed in matrix notation as:

$$I_p(w) = E(u')^2 + 2Fu'v' + G(v')^2$$

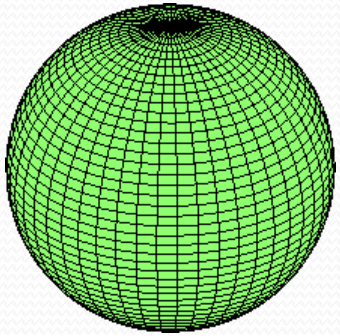
$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle_p$$
$$F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle_p$$
$$G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle_p$$

$$\Rightarrow \quad I_p(w) = \begin{pmatrix} u' & v' \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}$$

Wrap-up

The coefficients of the metric tensor are, in fact, *functions* defined on the surface element. Recall from this example:

$$\mathbf{x} : (0, 2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}^3 \qquad \mathbf{x}(u, v) = \begin{pmatrix} \cos(u)\cos(v) \\ \sin(u)\cos(v) \\ \sin(v) \end{pmatrix}$$



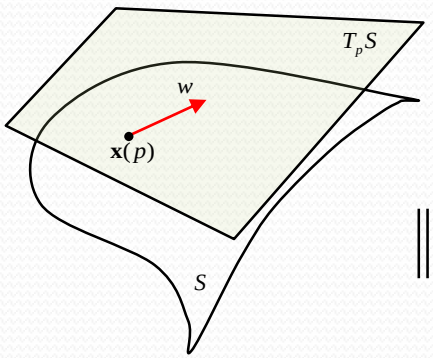
$$g = \mathrm{D}\mathbf{x}^{\mathrm{T}}\mathrm{D}\mathbf{x} = \begin{pmatrix} \cos^2(v) & 0 \\ 0 & 1 \end{pmatrix}$$

Thus, if $w = \alpha\mathbf{x}_u + \beta\mathbf{x}_v$ is the tangent vector to the sphere at point $\mathbf{x}(u,v)$, then its squared length is given by $|w|^2 = I(w) = \alpha^2\cos^2(v) + \beta^2$.

Wrap-up

Length of a tangent vector

$w \equiv (\alpha, \beta)$

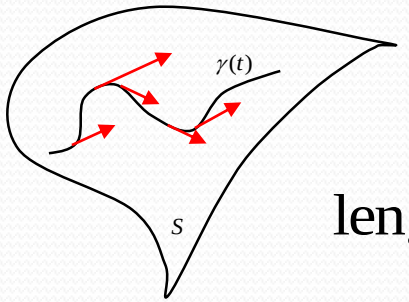


$$\|w\|^2 = \begin{pmatrix} \alpha & \beta \end{pmatrix} g \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Length of a curve

$\gamma(t) = \mathbf{x}(u(t), v(t))$

$\gamma : (0, T) \to S$



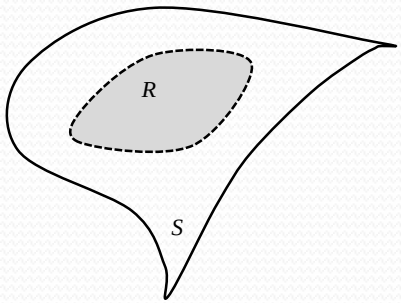
$$\text{length}(\alpha) = \int_0^T \sqrt{\begin{pmatrix} u' & v' \end{pmatrix} g \begin{pmatrix} u' \\ v' \end{pmatrix}}\, dt$$

$$\text{length}(\alpha) = \int_\alpha ds$$

Wrap-up

Area of a region

$Q = \mathbf{x}^{-1}(R)$

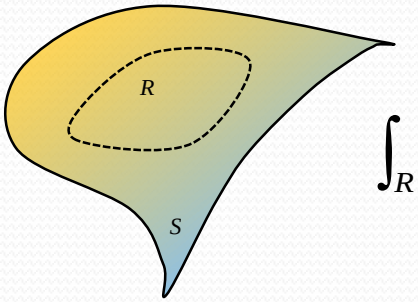


$$A(R) = \iint_Q \sqrt{\det g}\, du dv$$

$$A(R) = \int_R da$$

Integral of a function

$f : S \to \mathbf{R}$

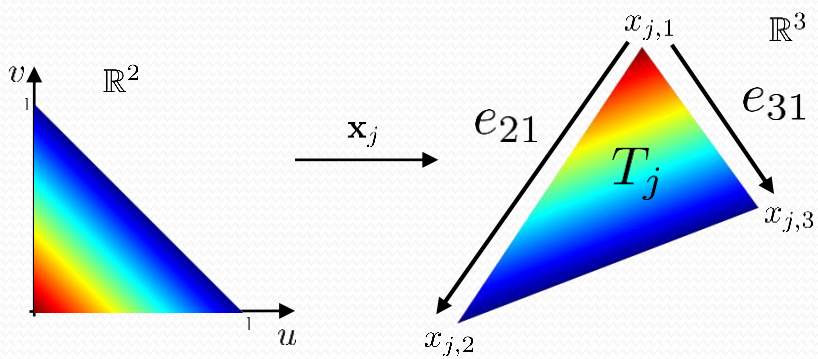


$$\int_R f(x)dx = \iint_Q f(\mathbf{x}(u,v))\sqrt{\det g}\, du dv$$

Wrap-up

When we have triangle meshes, discretizing all these quantities is quite straightforward.

$$\mathbf{x}_j(u,v) = x_{j,1} + u(x_{j,2} - x_{j,1}) + v(x_{j,3} - x_{j,1})$$



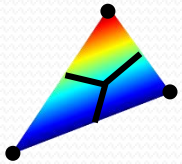
$$\mathbf{x}_u = x_{j,2} - x_{j,1} = e_{21}$$

$$\mathbf{x}_v = x_{j,3} - x_{j,1} = e_{31}$$

$$g_j = \begin{pmatrix} \|e_{21}\|^2 & \langle e_{21}, e_{31}\rangle \\ \langle e_{21}, e_{31}\rangle & \|e_{31}\|^2 \end{pmatrix}$$

$$\int_{T_j} f \; da = \int_0^1 \int_0^{1-u} f(\mathbf{x}(u,v)) \sqrt{\det g_j} \, du dv$$

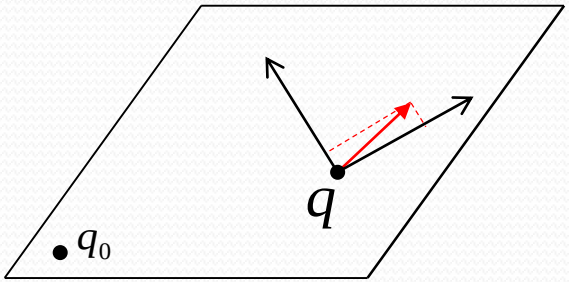
$$= \tfrac{1}{3}(f(x_{j,1}) + f(x_{j,2}) + f(x_{j,3}))A(T_j)$$

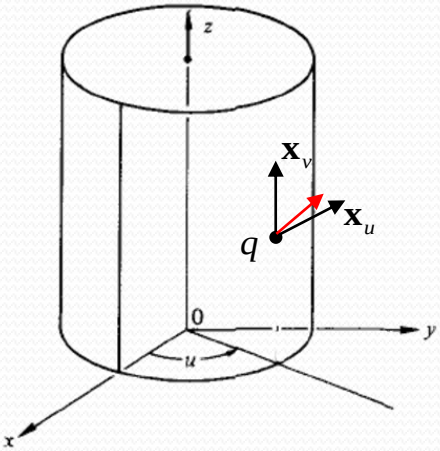


Local isometries

We have already seen that plane and cylinder behave locally in the same way, since their metric tensors are equal (at least on the surface elements we considered).

We captured this behavior by saying that plane and cylinder are «locally isometric». We will now give a more formal definition for **isometry**, and we will link it to the notion we already have from metric geometry.

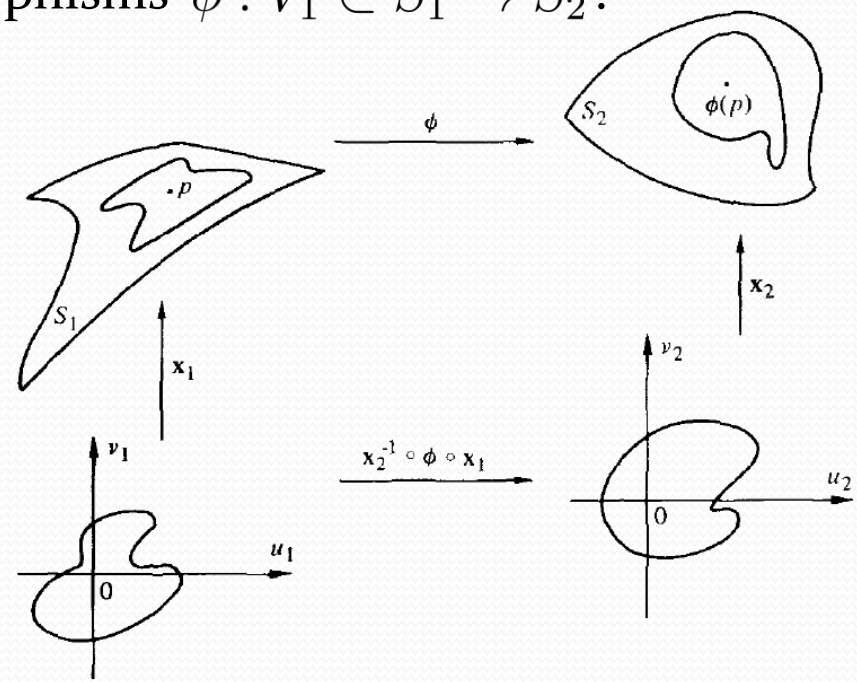




Maps between manifolds

Until now we have been mostly considering individual shapes and their associated charts $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$.

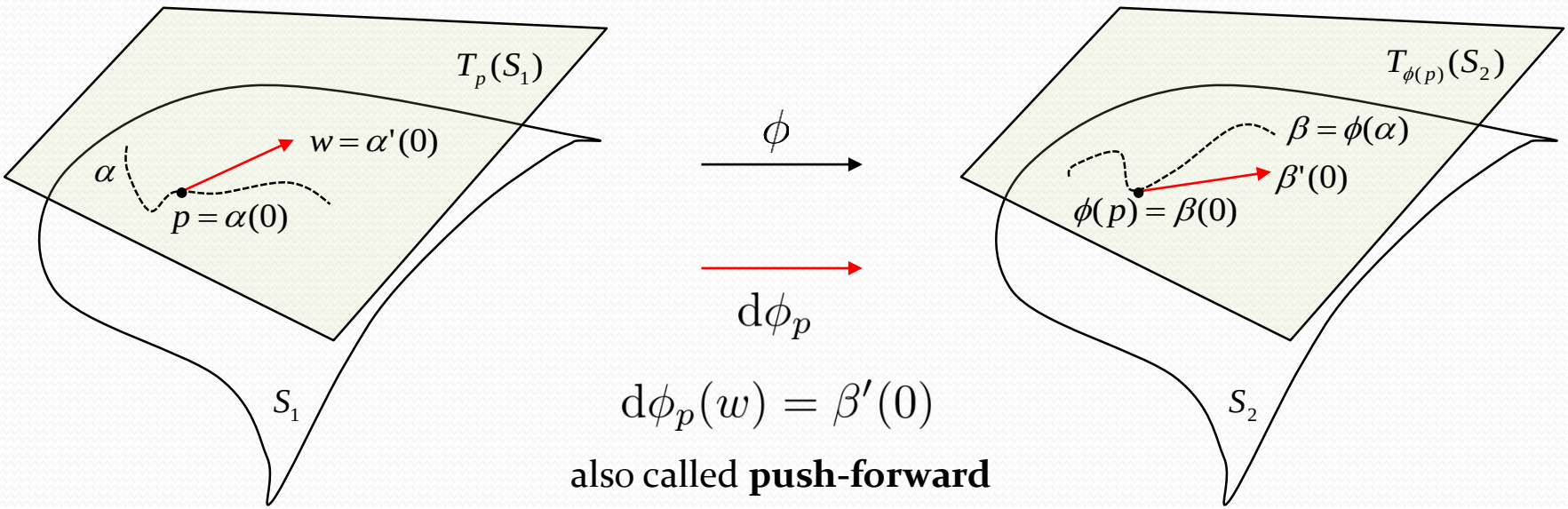
We can actually transfer this knowledge to **maps between manifolds**, namely diffeomorphisms $\phi : V_1 \subset S_1 \to S_2$.



ϕ is differentiable at p if $\mathbf{x}_2^{-1} \circ \phi \circ \mathbf{x}_1$ is differentiable at $q = \mathbf{x}_1^{-1}(p)$

Differential of a map

We can define the differential of a map between manifolds, as we did with charts. Given a map $\phi : V_1 \subset S_1 \to S_2$, the differential will be a linear map $\mathrm{d}\phi_p : T_p(S_1) \to T_{\phi(p)}(S_2)$ which maps tangent vectors to tangent vectors:

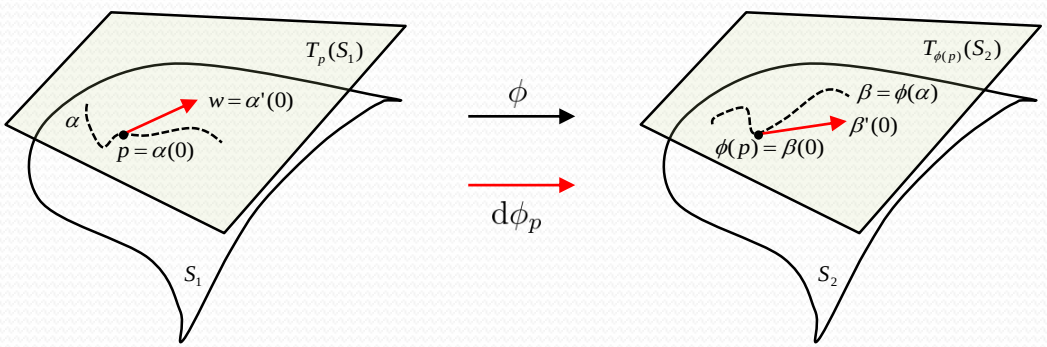


$$\mathrm{d}\phi_p(w) = \beta'(0)$$

also called **push-forward**

Differential of a map

The vector $\mathrm{d}\phi_p(w) = \beta'(0)$ does not depend on the choice of α, and the differential map $\mathrm{d}\phi_p : T_p(S_1) \to T_{\phi(p)}(S_2)$ is linear:



$$\beta(t) = \phi(\alpha(t)) = \phi(u(t), v(t))$$

Here for simplicity we are calling *u*, *v* the local basis on the tangent plane $T_p(S_1)$

Thus, we actually have $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ and we have the representation:

$$\phi(u, v) = (\phi_1(u, v), \phi_2(u, v))$$

Let us compute $\beta'(0)$ by applying the chain rule:

$$\beta' = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial t} = \frac{\partial \phi}{\partial u}u' + \frac{\partial \phi}{\partial v}v' \Rightarrow \beta'(0) = \mathrm{d}\phi_p(w) = \underbrace{\begin{pmatrix} \frac{\partial \phi_1}{\partial u} & \frac{\partial \phi_1}{\partial v} \\ \frac{\partial \phi_2}{\partial u} & \frac{\partial \phi_2}{\partial v} \end{pmatrix}}_{\text{Jacobian of } \phi} \begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix}$$

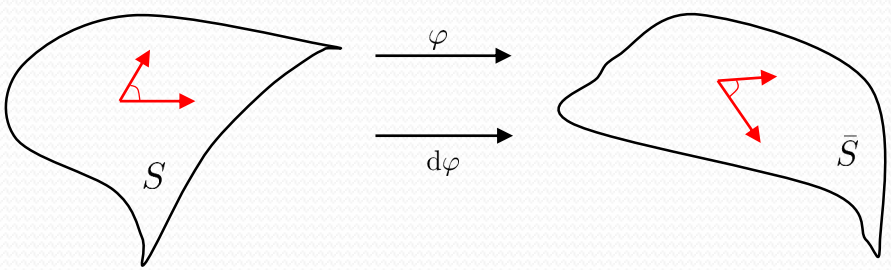
Isometries

A diffeomorphism $\varphi : S \to \bar{S}$ is called an **isometry** if

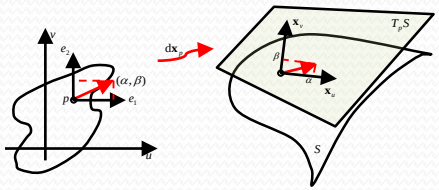
$$\langle w_1, w_2 \rangle = \langle \mathrm{d}\varphi_p(w_1), \mathrm{d}\varphi_p(w_2) \rangle$$

for all $p \in S$ and all pairs of tangent vectors $w_1, w_2 \in T_p S$.

In other words, a diffeomorphism φ is an isometry if its associated differential $\mathrm{d}\varphi$ preserves the inner product.



Remember when we said:



«isometric parametrization» $F \equiv 0\,, E = G = 1$

Isometries

A direct consequence is that <u>isometries preserve the first fundamental form</u>:

$$I_p(w) \underset{\substack{\uparrow \\ \text{first} \\ \text{fundamental} \\ \text{form on } S}}{=} \langle w, w \rangle \underset{\substack{\uparrow \\ \text{isometry}}}{=} \langle \mathrm{d}\varphi_p(w), \mathrm{d}\varphi_p(w) \rangle \underset{\substack{\uparrow \\ \text{first} \\ \text{fundamental} \\ \text{form on } \bar{S}}}{=} I_{\varphi(p)}(\mathrm{d}\varphi_p(w))$$

for all $w \in T_pS$.

The converse is also true. If a diffeomorphism φ preserves the first fundamental form, then it is an isometry:

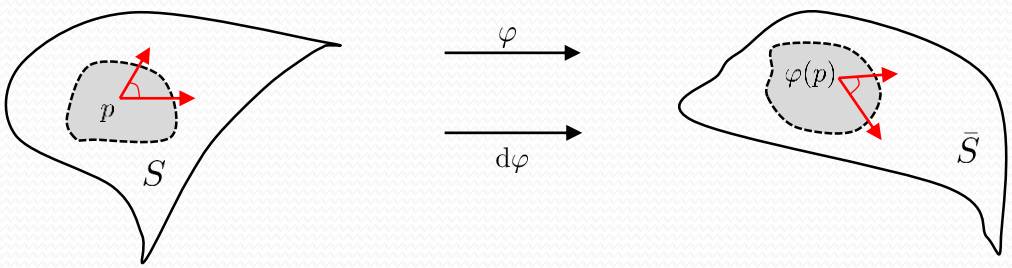
$$\begin{aligned} 2\langle w_1, w_2 \rangle &= I_p(w_1 + w_2) - I_p(w_1) - I_p(w_2) \\ &= I_{\varphi(p)}(\mathrm{d}\varphi_p(w_1 + w_2)) - I_{\varphi(p)}(\mathrm{d}\varphi_p(w_1)) - I_{\varphi(p)}(\mathrm{d}\varphi_p(w_2)) \\ &= 2\langle \mathrm{d}\varphi_p(w_1), \mathrm{d}\varphi_p(w_2) \rangle \end{aligned}$$

by assumption that φ preserves the first fundam. form

Local isometries

Note that the definition we gave for isometry <u>requires φ to be a diffeomorphism</u>.

If this is not the case, then a map $\varphi : V \to \bar{S}$ of a neighborhood V of $p \in S$ is called a **local isometry** at p if there exists a neighborhood $\bar{V}$ of $\varphi(p) \in \bar{S}$ such that $\varphi(p) : V \to \bar{V}$ is an isometry.

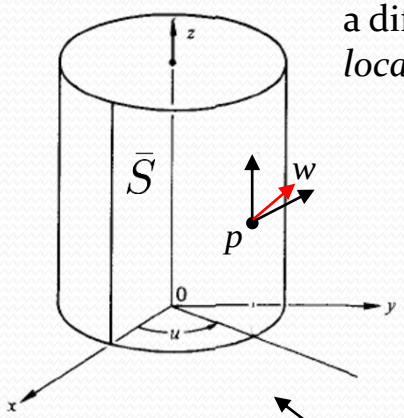


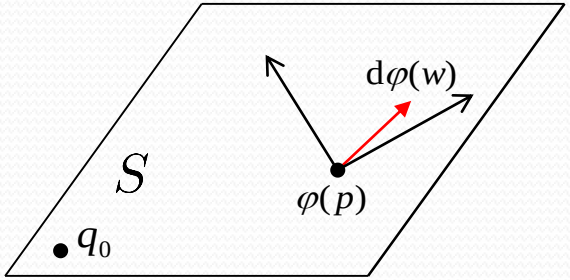
If there exists a local isometry at every $p \in S$, the surface S is said to be **locally isometric** to $\bar{S}$.

Example

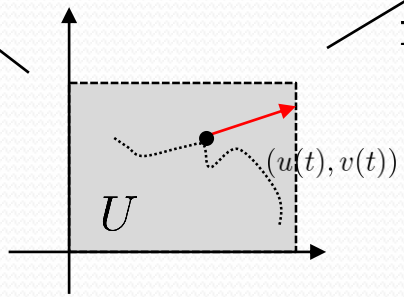
Note that $\varphi : \bar{\mathbf{x}}(U) \to \mathbf{x}(U)$ is indeed a diffeomorphism, but it is acting *locally*.

$$\varphi = \mathbf{x} \circ \bar{\mathbf{x}}^{-1}$$





$\bar{\mathbf{x}}$ $\mathbf{x}$



$$w = \bar{\mathbf{x}}_u u' + \bar{\mathbf{x}}_v v'$$

w is tangent to the curve $\bar{\mathbf{x}}(u(t), v(t))$

$$\mathrm{d}\varphi(w) = \mathbf{x}_u u' + \mathbf{x}_v v'$$

$\mathrm{d}\varphi(w)$ is tangent to the curve $\varphi(\bar{\mathbf{x}}(u(t), v(t))) = \mathbf{x}(u(t), v(t))$

As we have seen previously, we can get the same metric tensor for $\bar{\mathbf{x}}$ and $\mathbf{x}$. Thus, the two surfaces are locally isometric, since it holds:

$$I_p(w) = \bar{E}(u')^2 + 2\bar{F}u'v' + \bar{G}(v')^2 = E(u')^2 + 2Fu'v' + G(v')^2 = I_{\varphi(p)}(\mathrm{d}\varphi_p(w))$$

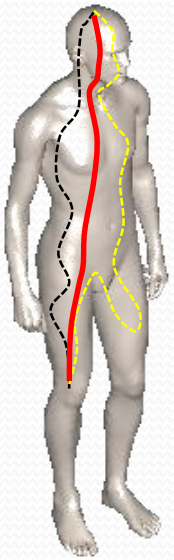
Intrinsic distance

We have seen how to use the first fundamental form to measure lengths of paths on a surface. This allows us to introduce a notion of «intrinsic» distance for points on the surface.

We define the distance $d(p,q)$ between two points of S as

$$d(p,q) = \inf_{\alpha:[0,1]\to S} \int_0^1 \|\alpha'(t)\| dt$$

where $\alpha(0) = p,\ \alpha(1) = q\,.$



According to this definition, every regular surface comes with a «natural» metric induced by the first fundamental form (the fact that d defined above is actually a metric should be proven, but we will not do it here).

Isometries: equivalence of the definitions

The distance *d* is invariant under isometries, since isometries preserve the first fundamental form. That is, if $\varphi : S \to \bar{S}$ is an isometry, then

$$d(p,q) = d(\varphi(p), \varphi(q))$$

for all $p, q \in S$.

From this proposition it seems like our original notion of isometry (i.e. from the point of view of metric spaces) is just a consequence of the new «differential» definition we gave in the previous slides.

In fact, we will now show that *the two definitions are equivalent* if we consider the natural, intrinsic metric induced by the first fundamental form.

Equivalence (1/2)

$$d(p,q) = \inf_{\alpha:[0,1]\to S} \int_0^1 \|\alpha'(t)\| dt = \inf_{\alpha:[0,1]\to S} \int_0^1 \sqrt{I_p(w(t))} dt$$

If $\varphi : S \to \bar{S}$ is an isometry, then $I_p(w) = I_{\varphi(p)}(\mathrm{d}\varphi_p(w))$ and thus by integrating we get:

$$\int_0^1 \sqrt{I_p(w(t))} dt = \int_0^1 \sqrt{I_{\varphi(p)}(\mathrm{d}\varphi_p(w(t)))} dt$$

In particular, the infimum will also have the same value. As a consequence,

$$\inf_{\alpha:[0,1]\to S} \int_0^1 \|\alpha'(t)\| dt = \inf_{\varphi\circ\alpha:[0,1]\to \bar{S}} \int_0^1 \|(\varphi\circ\alpha)'(t)\| dt$$

this is a little shaky, but it gives the idea...

$$\Downarrow$$

$$d(p,q) = d(\varphi(p), \varphi(q))$$

Equivalence (2/2)

Let us now assume that $\varphi : S \to \bar{S}$ is such that, for all pairs of points *p*,*q*:

$$d(p,q) = d(\varphi(p), \varphi(q))$$

We wish to prove that φ is an isometry, that is:

$$I_p(w) = I_{\varphi(p)}(\mathrm{d}\varphi_p(w))$$

The proof is not obvious, and there are a few things one should take care of:

- Existence of length-minimizing regular curves (Hopf-Rinow theorem)
- We already know that distance preservation implies continuity, but does it also imply smoothness?

The fact that "metric isometry" implies "differential isometry" is stated in the Myers-Steenrod theorem.

Suggested reading

- *Differential geometry of curves and surfaces*. Do Carmo – Chapters 2.4, 4.1, 4.2, Exercises 2, 3, 9, 18