Analysis of Three-Dimensional Shapes
(IN2238, TU München, Summer 2015)

Laplacian spectrum and eigenfunctions
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Seminar

«Scale-invariant heat kernel signatures»

Matthias Ebert

Thursday, May 21th

14:00 Room 02.09.023

Seminar

Dense non-rigid shape correspondence using random forests

Lucas Weidner

Thursday, May 21th
14:00 Room 02.09.023

Wrap-up

We have studied the **divergence theorem**, and seen how to obtain an expression in **local coordinates** for the divergence in the case of a manifold.

$$\langle \nabla f, \vec{V} \rangle = -\langle f, \mathrm{div}\vec{V} \rangle$$

In particular, we first rewrote the two inner products in local coordinates:

$$-\langle \nabla f, \vec{V} \rangle = \int_U \tilde{f}(x) \sum_i \frac{\partial}{\partial x_i} (\sqrt{\det g}\tilde{\vec{V}}_i(x))dx \qquad \langle f, \mathrm{div}\vec{V} \rangle = \int_U \tilde{f}(x)\mathrm{div}\tilde{\vec{V}}(x)\sqrt{\det g}dx$$

And then solved for the divergence, yielding the expression:

$$\mathrm{div}\vec{V}(x) = \frac{1}{\sqrt{\det g}} \sum_i \frac{\partial}{\partial x_i} \left(\vec{V}_i(x) \sqrt{\det g} \right)$$

Wrap-up

Replacing $\vec{V} = \nabla v$ in the divergence theorem, we got the expression:

$$-\langle \nabla f, \nabla v \rangle = \langle f, \mathrm{div} \nabla v \rangle$$

The operator $\Delta := \mathrm{div} \circ \nabla$ is called the **Laplace-Beltrami operator**.

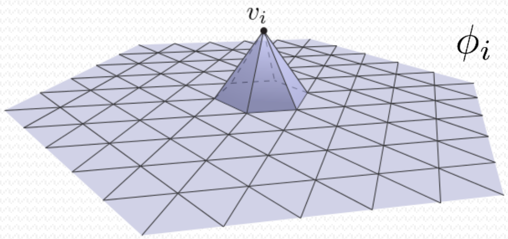
The expression in local coordinates for Δ can now be easily obtained as:

$$\Delta f = \frac{1}{\sqrt{\det g}} \sum_{i,j} \frac{\partial}{\partial x_i} \left(g^{ij} \frac{\partial \tilde{f}(x)}{\partial x_j} \sqrt{\det g} \right)$$

From which it is clear that the operator **only depends on the metric *g***.

Wrap-up

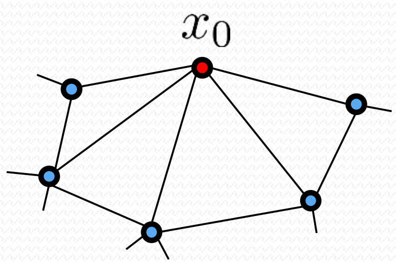
We discretized the Laplace-Beltrami operator using **FEM**. Given a finite element basis $\{\phi_j\}_{j=1,\ldots,n}$, we wrote the weak relation:



$$\langle h, \phi_j \rangle = \langle \Delta f, \phi_j \rangle$$

The two sides of the equation can be rewritten as $Mh = Cf$, hence giving us:

$$L = M^{-1}C$$



Intuitively, the Laplacian acts in such a way that it provides, for each mesh point, the **difference** of the function computed at that point **with the average** of the function at its 1-ring neighborhood.

Wrap-up

The stiffness and mass matrices are easy to compute for any given mesh:

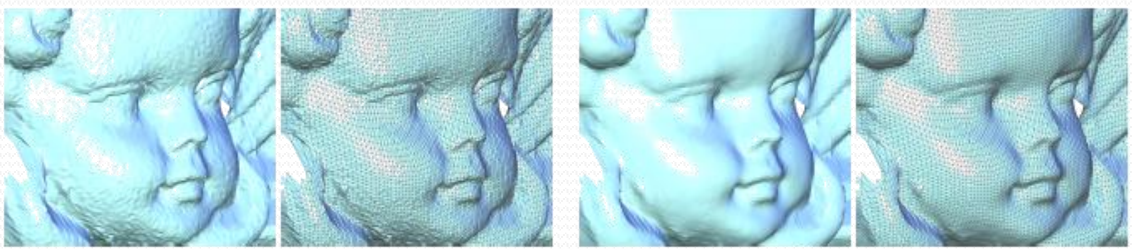
$$C_{ij} = \begin{cases} \dfrac{\cot\alpha_{ij} + \cot\beta_{ij}}{2} & (i, j) \text{ edge} \\ -\sum_{k \in N(i)} C_{ik} & i = j \end{cases}$$

$$M_{ij} = \begin{cases} \dfrac{A(T_1) + A(T_2)}{12} & (i, j) \text{ edge} \\ \dfrac{\sum_{k \in N(i)} A(T_k)}{6} & i = j \end{cases}$$

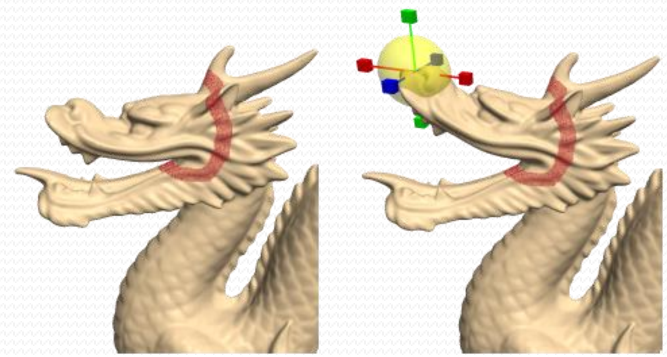
T1, T2 are the triangles that share the edge (*i*,*j*)

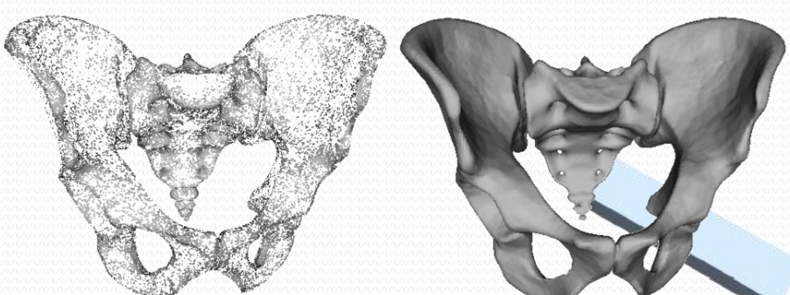
Laplacian-based geometry processing

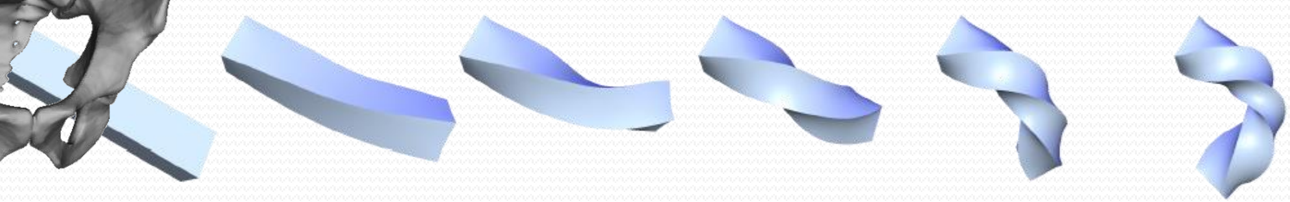
In this course we are mostly interested in the **analysis** of shapes rather than their processing. However, there is a large body of work dedicated to mesh **processing, modeling and reconstruction** based on the Laplacian.



(a) original (b) smoothed







source $t = 0.2$ $t = 0.4$ $t = 0.6$ $t = 0.8$ target

Laplacian on a surface

$$-\langle \nabla f, \nabla v \rangle = \langle f, \mathrm{div} \nabla v \rangle$$

Rewriting the divergence theorem in terms of Δ yields the **Green's identities**:

$$\int_S f \Delta v = -\int_S \langle \nabla f, \nabla v \rangle = \int_S v \Delta f$$

From the above we see that the Laplacian is a **self-adjoint operator**, since we have $\langle f, \Delta v \rangle = \langle \Delta f, v \rangle$.

Notice that the above relationships *only hold for manifolds without boundary*. Otherwise, additional integration terms on the boundary are also present!

Spectral theorem on a Hilbert space

Let us consider a Hilbert space *V* and a Hermitian map *A* on *V*. That is:

$$\langle Av, w \rangle = \langle v, Aw \rangle \quad \forall v, w \in V$$

Note that if *A* is a real matrix, this is equivalent to $A^\top = A$.

Now consider the eigenvalue problem:

$$Av = \lambda v$$

It is easy to show (do it!) that all eigenvalues of *A* are **real**. Further, eigenvectors associated to distinct eigenvalues are **orthogonal** (and can of course be chosen to be **orthonormal**).

Theorem: The eigenvectors of *A* form an orthonormal basis of *V*.

Helmholtz equation

The eigen-decomposition of the Laplace-Beltrami operator gives rise to the so-called **Helmoltz equation**:

$$\Delta f = -\lambda f$$

Since the Laplace operator is self-adjoint, its matrix representation is a Hermitian matrix. Then, we already know that its eigenvalues are real, and that the **eigenfunctions** f are orthogonal and form a basis.

In particular, we are looking at the linear operator $\Delta : \mathcal{F}(S) \to \mathcal{F}(S)$, which maps scalar functions defined on the manifold *S* to scalar functions on *S* itself.

Thus, the eigenfunctions of Δ form a basis for the function space $\mathcal{F}(S)$.

Spectral theorem in our case

The spectral theorem actually tells us that the eigenfunctions of Δ form an **orthonormal** basis for functions defined on the manifold.

Clearly, orthonormality here is understood with respect to the manifold inner product $\langle f, g \rangle_S$.

In fact, let $\Delta \phi_1 = \lambda_1 \phi_1$ and $\Delta \phi_2 = \lambda_2 \phi_2$, with $\lambda_1 \neq \lambda_2$. Then Green's formulas tell us:

$$\langle \phi_1, \Delta \phi_2 \rangle = \langle \phi_2, \Delta \phi_1 \rangle \Rightarrow \lambda_2 \langle \phi_1, \phi_2 \rangle = \lambda_1 \langle \phi_2, \phi_1 \rangle \Rightarrow \langle \phi_1, \phi_2 \rangle = \langle \phi_2, \phi_1 \rangle = 0$$

Eigen-decomposition in practice

Note that, since $h = M^{-1}Cf = Lf$, we can rewrite the Helmholtz equation as an equivalent *generalized* eigenvalue problem:

$$\Delta f = \lambda f \Rightarrow M^{-1}Cf = \lambda f \Rightarrow \boxed{Cf = \lambda M f}$$

The eigenvalues are the same as in the original case. In particular, since *C* is symmetric and *M* is symmetric positive-definite, the generalized eigenvectors *f* are still **orthonormal** *with respect to the M-inner product*:

$$\langle f, g \rangle_M = f^{\mathrm{T}} M g$$

In other words, we are approximating the continuous inner product as follows:

$$\int_S f(x)g(x)dx \approx f^{\mathrm{T}} M g = \langle f, g \rangle_M$$

Integral of eigenfunctions

Consider Green's identities again:

$$\int_S f\Delta v = -\int_S \langle \nabla f, \nabla v \rangle = \int_S v \Delta f$$

Then it is easy to see that the following equality holds for any function *f*:

$$\int_S \Delta f = \int_S 1\Delta f = -\int_S \langle \nabla 1, \nabla f \rangle = 0$$

In particular, let $\Delta \phi_i = \lambda_i \phi_i$. Then we obtain:

$$\int_S \phi_i = \frac{1}{\lambda_i} \int_S \Delta \phi_i = 0$$

Dirichlet energy and eigenvalues

Let $\phi_i : S \to \mathbb{R}$ be an eigenfunction with corresponding eigenvalue λ_i, and consider the **Dirichlet energy** of ϕ_i:

$$\int_S \|\nabla \phi_i\|^2 = \int_S \langle \nabla \phi_i, \nabla \phi_i \rangle = -\int_S \phi_i \Delta \phi_i \; = \lambda_i \int_S \phi_i \; \phi_i$$

$$= \lambda_i$$

This provides us with a nice characterization of the eigenvalues, in terms of the corresponding eigenfunctions.

In particular, from the above relation we see that if $\lambda_i = 0$, then ϕ_i must be a **constant** function. Further, $\lambda_i = 0$ is always an eigenvalue of Δ, since $\Delta f = 0$ for any constant function *f*.

Rayleigh quotient and eigenvalues

Consider again the Dirichlet energy of the eigenfunctions:

$$\int_S \|\nabla\phi\|^2 = -\int_S \phi\Delta\phi$$

In matrix notation, we can write this energy term as $\phi^\top L\phi$, and then consider the constrained minimization problem:

$$\min_{\phi:S\to\mathbb{R}} \phi^\top L\phi$$

$$\text{s.t. } \|\phi\|^2 = 1 \quad \text{and} \quad \sum_i \phi(x_i) = 0$$

As we know from Rayleigh's theorem, this is minimized by the eigenvector of *L* corresponding to the minimum, non-zero eigenvalue. That is, it is minimized by **the first non-constant eigenfunction of** Δ.

First non-constant eigenfunction

Let us rewrite the objective function as a sum:

$$\min_{\phi:S\to\mathbb{R}} \phi^\top L\phi = \sum_{i,j} w_{ij}(\phi(x_i) - \phi(x_j))^2$$

$$\text{s.t. } \|\phi\|^2 = 1 \quad \text{and} \quad \sum_i \phi(x_i) = 0$$

Recall when we were looking for Euclidean embeddings of a manifold:

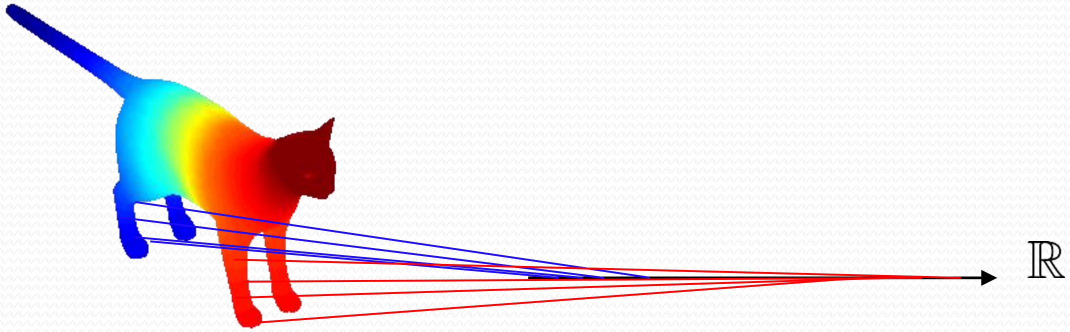
$$\min_{\phi:S\to\mathbb{R}^m} \sum_{i,j} |d_S(x_i, x_j) - d_{\mathbb{R}^m}(\phi(x_i), \phi(x_j))|^2$$

Intuitively, supposing the weights w_{ij} correspond to edge lengths, the problem above is seeking **a one-dimensional embedding of the mesh on a line**, that tries to respect the edge lengths of the mesh. With respect to classical MDS, we are just considering a different stress function.

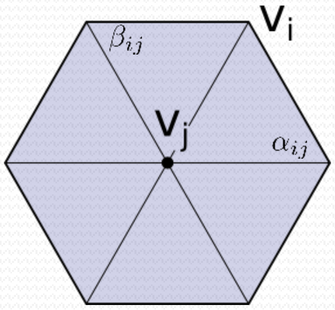
First non-constant eigenfunction

$$\min_{\phi: S \to \mathbb{R}} \phi^\top L \phi = \sum_{i,j} w_{ij} (\phi(x_i) - \phi(x_j))^2$$

$$\text{s.t. } \|\phi\|^2 = 1 \quad \text{and} \quad \sum_i \phi(x_i) = 0$$



In graph analysis, the first non-constant eigenvector of the Laplacian is also called the **Fiedler vector**.

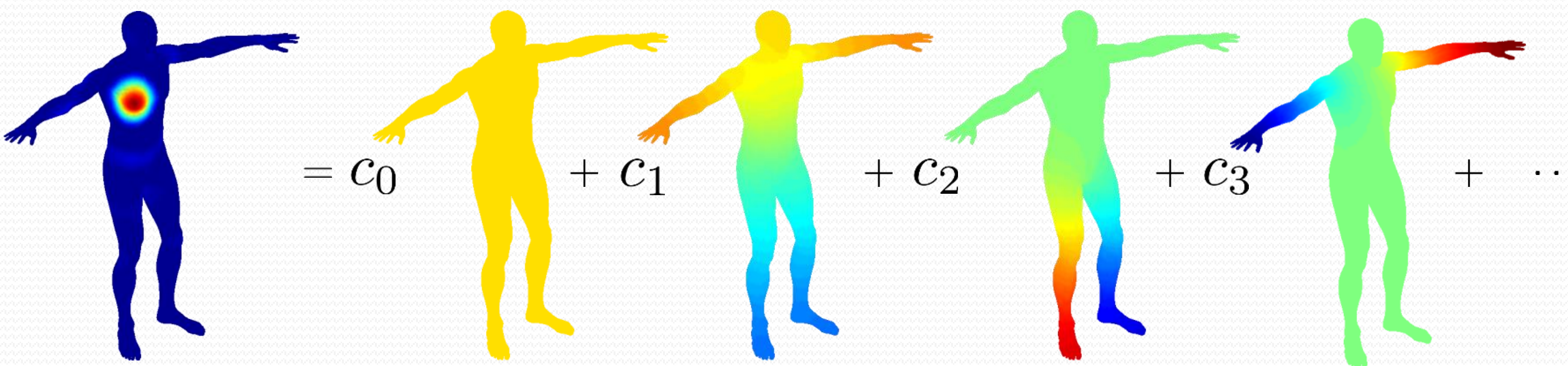


In our case (FEM), we have a weight which is proportional to the edge length:

$$w_{ij} = M_{ij}^{-1} \frac{\cot \alpha_{ij} + \cot \beta_{ij}}{2}$$

Spectral theorem

$$f = \sum_i c_i \phi_i \qquad \text{where } \phi_i \text{ are eigenfunctions of } \Delta$$



$$= c_0 \quad + c_1 \quad + c_2 \quad + c_3 \quad + \quad \cdots$$

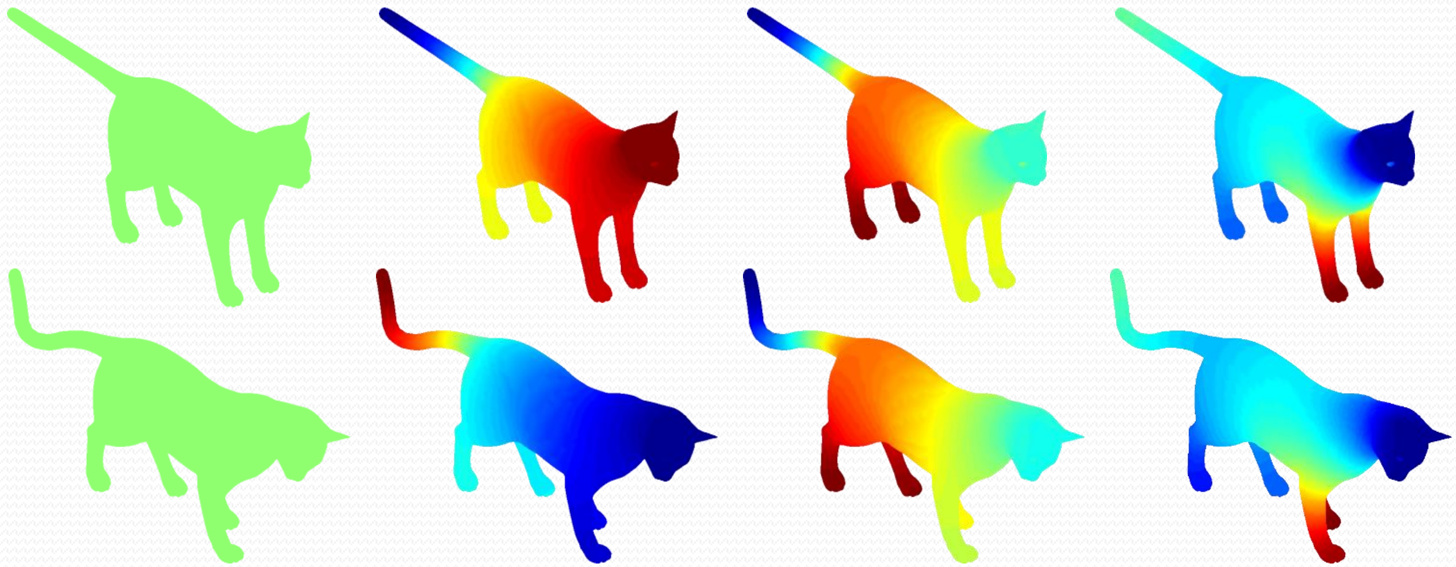
Since we are working with an orthonormal basis, the **Fourier coefficients** c_i can be recovered by the projections:

$$c_i = \langle f, \phi_i \rangle_S = \int_S f(x)\phi_i(x)dx$$

Invariance under isometries

$$\Delta f = \frac{1}{\sqrt{\det g}} \sum_{i,j} \frac{\partial}{\partial x_i} \left(g^{ij} \frac{\partial \tilde{f}(x)}{\partial x_j} \sqrt{\det g} \right)$$

Since the Laplace-Beltrami operator only depends on g, it is invariant under isometric deformations of the surface.



notice the sign flip here

Discrete spectrum

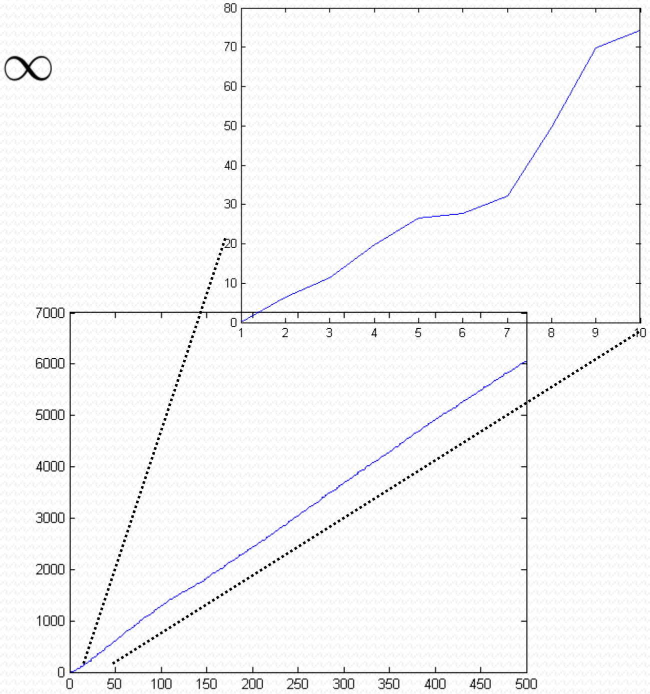
It can be shown that the **eigenvalues** of the Laplacian defined on a compact surface without boundary are **countable** with no limit-point except ∞, so we can order them:

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \to \infty$$

Another interesting relation is given by **Weyl's law**:

$$\lambda_j \sim \frac{\pi}{\int_S da} j \qquad \text{for } j \to \infty$$

Note that we can not say much about the multiplicity of eigenvalues.



Properties of the spectrum

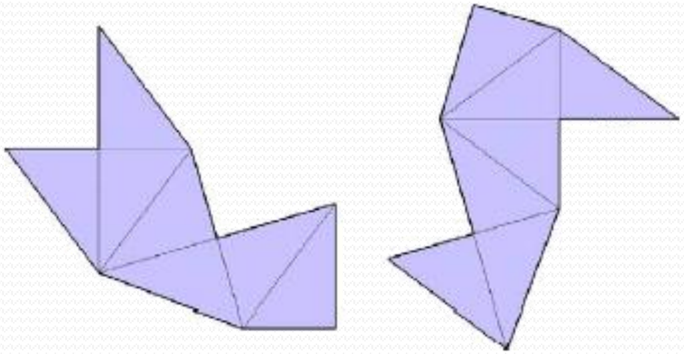
The spectrum of the Laplacian has several interesting properties for the purposes of shape analysis:

- It has a **canonical ordering** with respect to the Dirichlet energy of the eigenfunctions.
- It is an **isometry invariant** as it only depends on the metric.
- It allows to define **scale-invariant** properties of the shapes (more on this later).
- It depends **continuously** on the Riemannian metric of the manifold.
- It is easy and **efficient** to compute.
- Important **information on the shape** can be extracted from the spectrum alone (for example surface area, topological properties, symmetries, etc.)

Shape-DNA

Unfortunately, the spectrum does not completely determine the shape of the underlying manifold, even though geometrical data is contained in the eigenvalues.

This means that, unfortunately, one cannot «hear the shape of the drum».

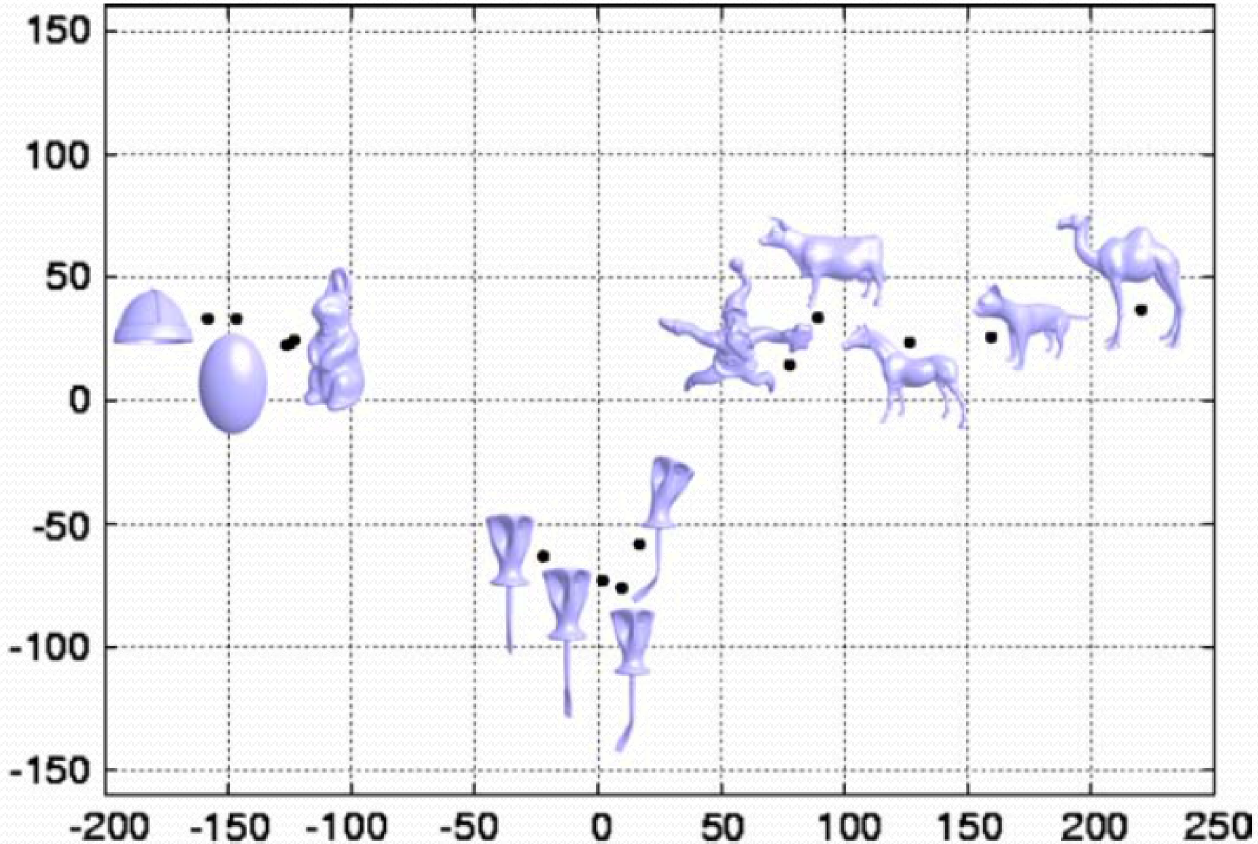


non-isometric, but isospectral domains

Nevertheless, examples like this are very difficult to construct and are understood to be relatively rare phenomena.

The Laplacian spectrum used as a **global descriptor** for a shape is known as the **Shape DNA**.

Shape-DNA



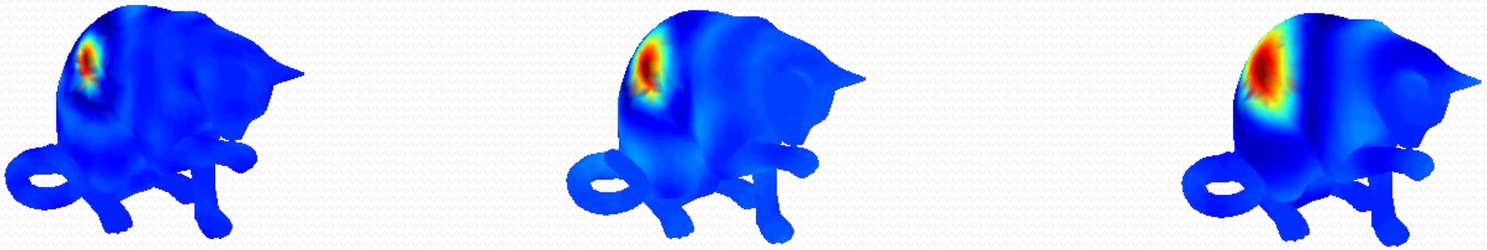
Function approximation

In general, in order to construct a scalar function using the linear combination given by $f = \sum c_i \phi_i$, we should use all eigenfunctions. In the discrete case there are *n* of them, where *n* is the number of points in the mesh.

However, the spectrum ordering gives us a natural scale space. In particular, if we truncate the summation to the first *k* terms, we get an approximation:

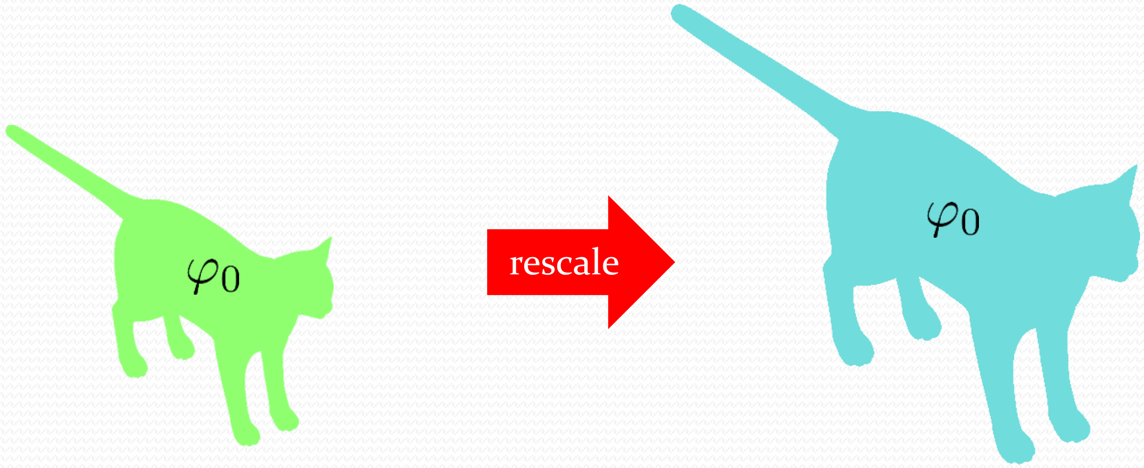
$$\sum_{i=1}^{k} c_i \phi_i \approx \sum_{i=1}^{\infty} c_i \phi_i$$

this is understood in the L2 sense



$k = 200$ $k = 100$ $k = 50$

Changes in scale



What happens to the eigenvalues and eigenfunctions when we simply rescale a shape?

Weyl's law is already suggesting us that something is going to change.

First eigenfunction (constant)

Observe that for φ_0 we have:

$$\langle \varphi_0, \varphi_0 \rangle_M = \int_S K \cdot K dx = K^2 \int_S dx = K^2 |S|$$

$$\langle \varphi_0, \varphi_0 \rangle_M = 1 \qquad \text{by orthonormality of } \{\varphi_k\}$$

⇩

$$K = \frac{1}{\sqrt{|S|}}$$

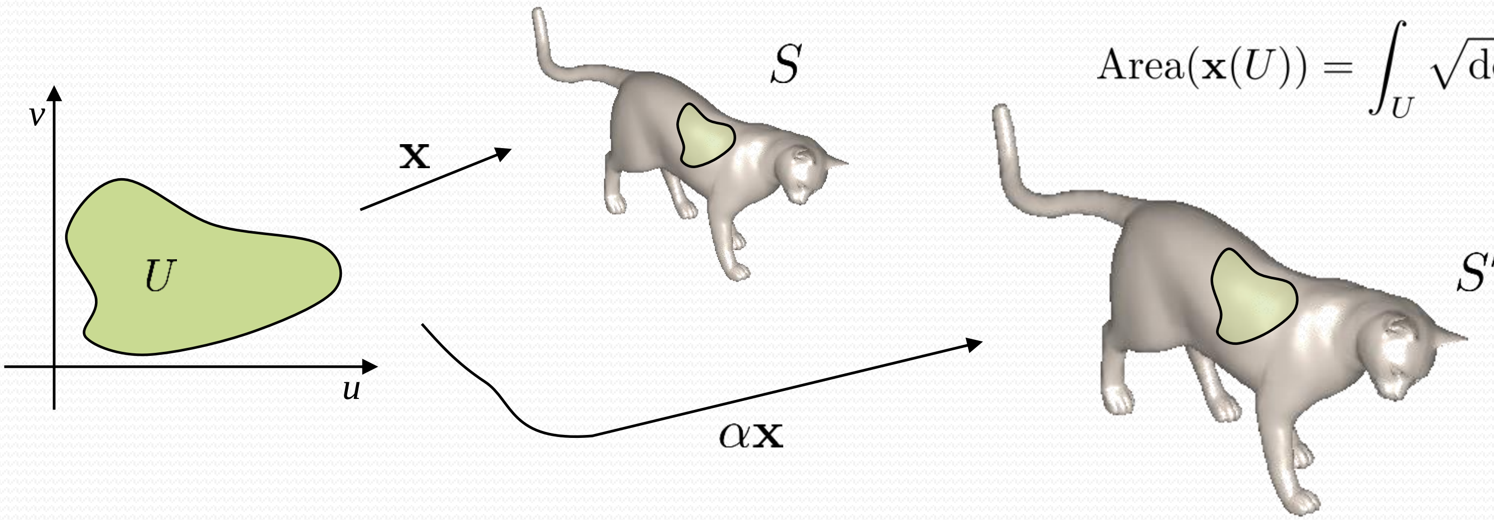
In general, the scale of the eigenfunctions *depends on the size of the shape*.

This should come as no surprise, remember for instance Weyl's law:

$$\lambda_j \sim \frac{\pi}{|S|} j \quad \text{for } j \to \infty$$

Rescaling areas

Let us be given a shape *S* and its scaled version $S' = \alpha S$



$$\text{Area}(\mathbf{x}(U)) = \int_U \sqrt{\det g}\, du dv$$

$$g' = \begin{pmatrix} \langle \alpha\mathbf{x}_u, \alpha\mathbf{x}_u \rangle & \langle \alpha\mathbf{x}_u, \alpha\mathbf{x}_v \rangle \\ \langle \alpha\mathbf{x}_v, \alpha\mathbf{x}_u \rangle & \langle \alpha\mathbf{x}_v, \alpha\mathbf{x}_v \rangle \end{pmatrix} = \alpha^2 g$$

$$\text{Area}(\alpha\mathbf{x}(U)) = \int_U \sqrt{\det g'}\, du dv = \int_U \sqrt{\det \alpha^2 g}\, du dv = \int_U \sqrt{\alpha^4 \det g}\, du dv = \alpha^2 \int_U \sqrt{\det g}\, du dv$$

$$= \alpha^2 \text{Area}(\mathbf{x}(U))$$

Rescaling eigenvalues

Let us be given a shape *S* and its scaled version $S' = \alpha S$, and let us consider the generalized eigenvalue problem for the first shape:

$$C\varphi = \lambda M \varphi$$

For the second shape, we have:

$$S \mapsto \alpha S$$
$$\lambda \mapsto \frac{1}{\alpha^2}\lambda$$

$$C\varphi' = \lambda' M' \varphi' = \lambda'(\alpha^2 M)\varphi' = (\alpha^2 \lambda') M \varphi'$$

$C' \equiv C$ since cotangents do not change with scale

the areas in *M* scale up with α^2

One could pre-process the shapes by normalizing their eigenvalues. For instance, pick an eigenvalue $\lambda^\star$ and rescale the given shape *S* as

$$S' = \sqrt{|\lambda^\star|}S$$

for example, choose $\lambda^\star = \max\{|\lambda_k|\}$

Rescaling eigenfunctions

What happens to the eigenfunctions?

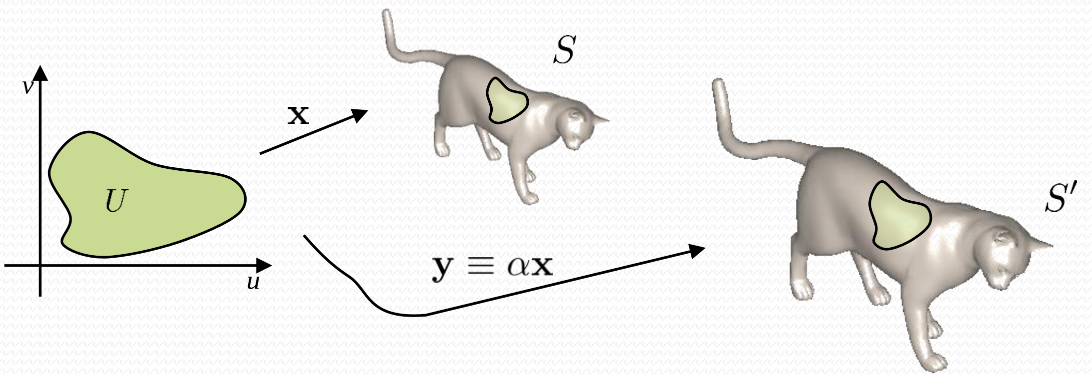
Let us have a look at what happens to the *first* (constant) eigenfunction:

$$\langle {\varphi'}_0, {\varphi'}_0 \rangle_{\alpha^2 M} = {K'}^2 \int_{\alpha S} dx = {K'}^2 |\alpha S| = {K'}^2 \alpha^2 |S|$$

$$K' = \frac{1}{\alpha \sqrt{|S|}}$$

It looks like eigenfunctions are rescaled as $\varphi \mapsto \frac{1}{\alpha}\varphi$. We are now going to prove this statement for arbitrary eigenfunctions.

Rescaling eigenfunctions



A few slides ago we showed that
$\mathrm{Area}(\alpha\mathbf{x}(U)) = \alpha^2 \mathrm{Area}(\mathbf{x}(U))$

In particular, since $y \equiv \alpha x$, we can compute the area element on *S'* as
$dy = \alpha^2 dx$

Let us consider the generic eigenfunction φ on *S*. How is it transformed by the rescaling $S \mapsto \alpha S$?

$$\varphi'(y) = \kappa \varphi\left(\frac{y}{\alpha}\right) \quad \text{unknown}$$

$$\int_{\alpha S} \varphi'^2(y) dy = \int_{\alpha S} \kappa^2 \varphi^2 \left(\frac{y}{\alpha}\right) dy = \kappa^2 \int_S \varphi^2(x)\alpha^2 dx = \alpha^2 \kappa^2 \int_S \varphi^2(x) dx = \alpha^2 \kappa^2$$

By orthonormality of φ': $\alpha^2 \kappa^2 = 1 \Rightarrow \kappa = \frac{1}{\alpha}$

Since this holds for any eigenfunction, we have proved that $\varphi \mapsto \frac{1}{\alpha}\varphi$

The Laplacian

Demo Time!