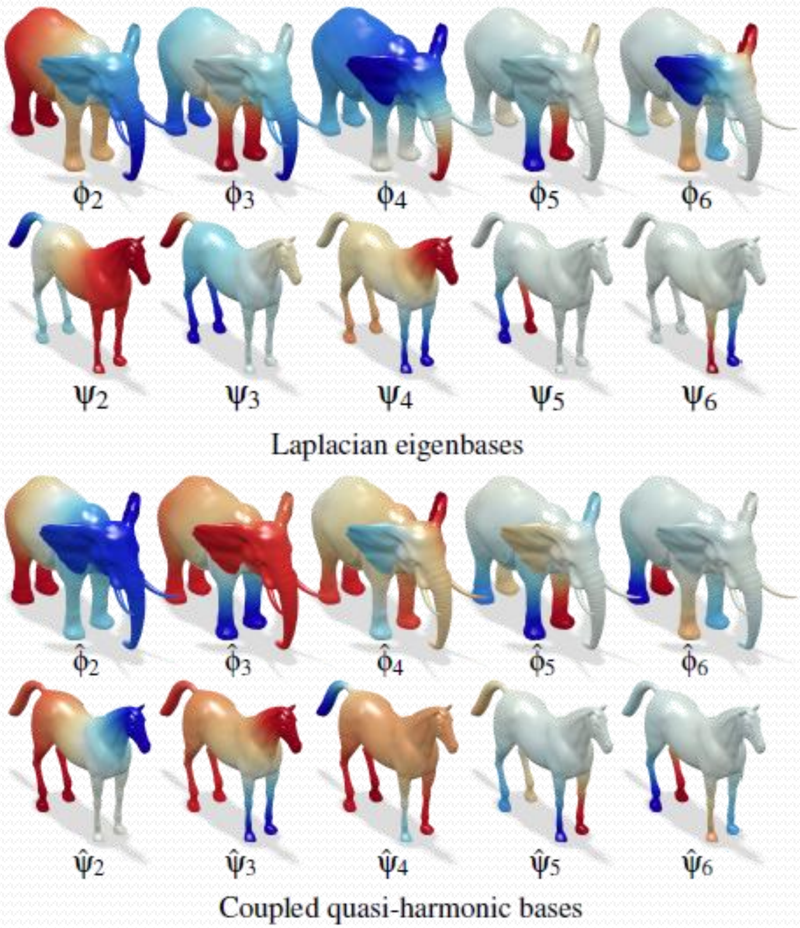
Analysis of Three-Dimensional Shapes
(IN2238, TU München, Summer 2015)

Shape differences
(15.06.2015)

Dr. Emanuele Rodolà
rodola@in.tum.de
Room 02.09.058, Informatik IX

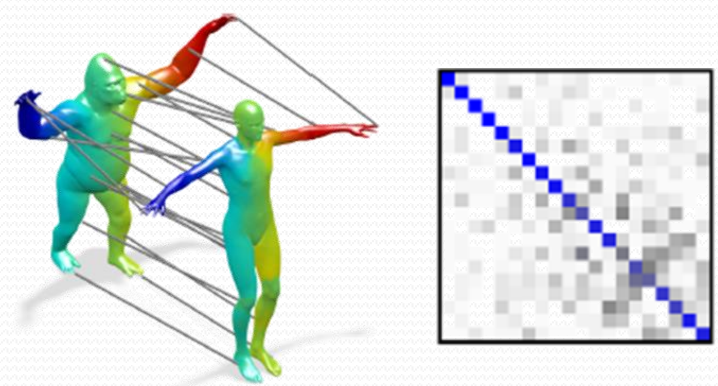
Wrap-up



We have introduced the notion of "compatible bases" as basis functions being related by the relation:

$$\phi_i = \pm\psi_i \circ T^{-1}$$

We have analyzed an algorithm to compute compatible bases among arbitrary shapes.

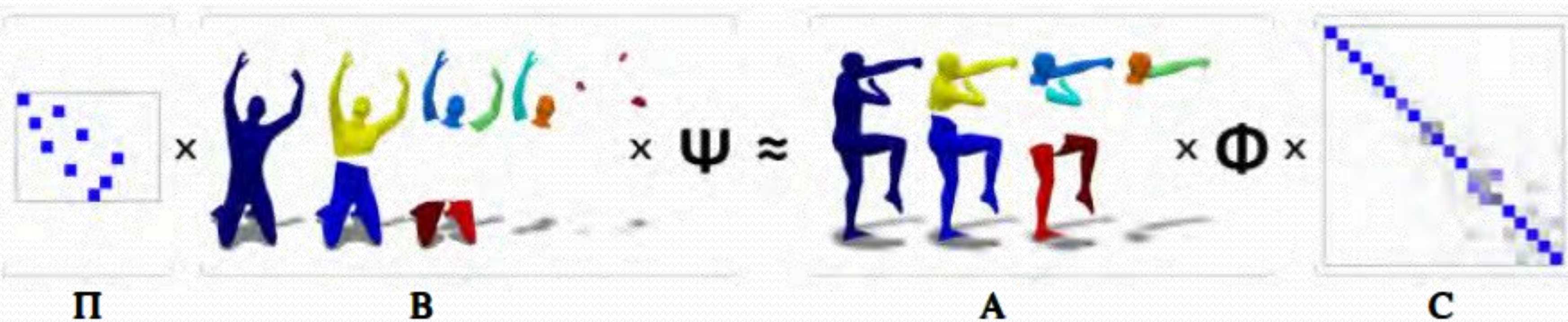


Wrap-up

Then, we have analyzed an extension to the functional maps framework, allowing to compute accurate correspondences with very sparse data:

$$\min_{C,\Pi} \|\Pi B - AC\|_F^2 + \lambda \|W \circ C\|_1$$

unknown permutation — unknown functional map — mask promoting diagonal C



Area preservation

In the following we will consider two shapes S_1 and S_2, and we will denote the associated quantities by using the same indices. For instance, M_1 is the mass matrix for S_1 , while da_2 is the local area element for S_2 .

We say that a bijection $T : S_2 \rightarrow S_1$ is **area-preserving** if it preserves the area elements:

$$da_2(x) = da_1(T(x)) \quad \forall x \in S_2$$

In the discrete case, we call a bijection / permutation "area-preserving" if and only if the area matrices are equal up to permutation:

$$M_1 = M_2$$

Area preservation

Suppose now that we are given a bijection $T : S_2 \to S_1$ such that:

$$\int_{S_1} f(x)g(x)da_1(x) = \int_{S_2} f(T(x))g(T(x))da_2(x)$$

for all scalar functions $f, g : S_1 \to \mathbb{R}$.

Then, if we define *f* and *g* to be indicator functions for some region $\Omega \subseteq S_1$, it follows that the area of Ω equals the area of $T^{-1}(\Omega)$.

In other words, we get that **if *T* preserves the inner product, then *T* is an area-preserving bijection**.

Area preservation

Conversely, suppose *T* is area preserving, that is $da_2(x) = da_1(T(x)) \quad \forall x \in S_2$

Then, for any function $h : S_1 \to \mathbb{R}$ we have:

$$\int_{S_1} h(x)da_1(x) = \int_{S_2} h(T(x))da_2(x)$$

By setting $h(x) = f(x)g(x)$, we get:

$$\int_{S_1} f(x)g(x)da_1(x) = \int_{S_2} f(T(x))g(T(x))da_2(x)$$

In other words, **if *T* is area-preserving then it preserves the inner product**.

Area preservation and inner products

Let us be given two surfaces S_1 and S_2 . Let $T : S_2 \to S_1$ be a bijection between them, and let $F : L^2(S_1) \to L^2(S_2)$ be the associated functional map, defined as:

$$F(f) = f \circ T$$

By writing the inner product as

$$h_a(f, g) = \int f(x)g(x)da(x)$$

the following holds **if and only if *T* is area preserving**:

$$h_a^1(f, g) = h_a^2(F(f), F(g))$$

Preserving areas is equivalent to preserving inner products.

Map distortion

$$h^1_a(f, g) = h^2_a(F(f), F(g))$$

It is natural to quantify the distortions induced by the map through the failure of this equality. This would give us a new measure of **shape similarity** in terms of **area distortion**.

Of course, such a notion will depend on the chosen functions *f* and *g*. The challenge is to provide a more *general* discrepancy measure.

Representation theorem

Let us recall the **Riesz representation theorem**:

H Hilbert space (vector space with an inner product)
This is our function space $L^2(S_1)$ with the usual manifold inner product $h_a^1(f,g)$

$H^* = \{\phi : H \to \mathbb{R} \mid \phi \text{ continuous , linear}\}$ dual space of *H*

In our case, let us just fix some function *g*, and consider $\phi = h_a(\cdot, g)$

Theorem: *Every* $\phi \in H^*$ can be written *uniquely* as an inner product:

$$\phi(y) = \langle y, x \rangle \quad \forall y \in H \quad \Rightarrow \quad h_a(f,g) = h_a(f,\underline{x})$$

unique

In particular, *x* can be written as the application of a *unique* linear transformation *D*. In other words, we can write:

$$h_a(f,g) = h_a(f, D(g))$$

Difference operator

$$h_a^1(f,g) = h_a^1(f, D(g))$$

This seems trivial, but now notice that, instead of computing $h_a^1(f,g)$ on S_1, we can apply the functional map F and compute instead $h_a^2(F(f), F(g))$.

Then, the representation theorem allows us to write:

$$\boxed{h_a^2(F(f), F(g)) = h_a^1(f, D(g))}$$

where $D : L^2(S_1) \to L^2(S_1)$ is a linear operator (in particular, it is a self-map) which we will call the **difference** between the two inner products.

Difference operator

$$h_a^2(F(f), F(g)) = h_a^1(f, D(g))$$

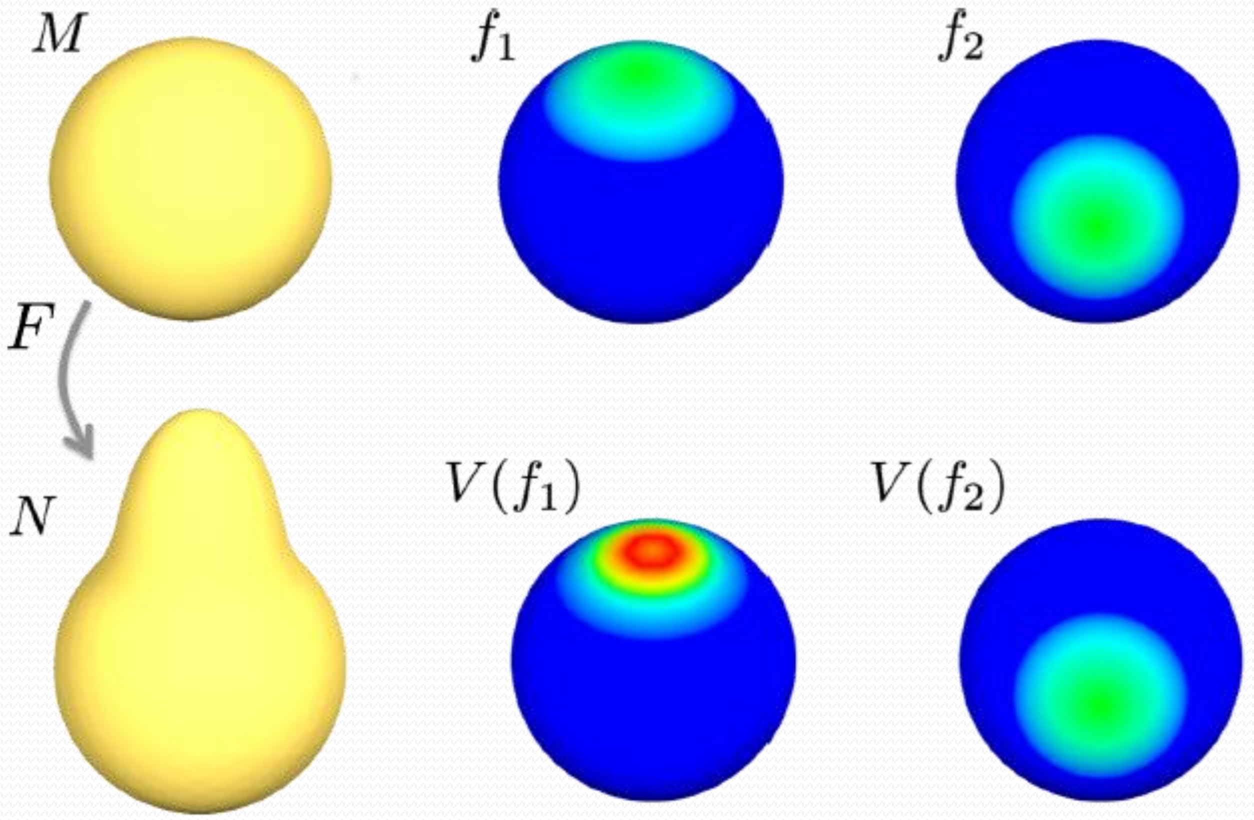
The linear operator *D* modifies the function *g*, so as to **exactly compensate for the area distortions** introduced by the functional map *F*.

The nice thing about this, is that *D* is "universal": it is unique and works simultaneously for all functions *f* and *g*.

The difference operator *D* depends on:

- The functional map *F*
- The inner products on the two shapes (in fact, we will see example of inner products which are *different* from the standard definition).

Difference operator: example



In this example, the difference operator is called *V* instead of *D*.

Shape difference

In the following we will rename the difference operator D simply as V_{S_1,S_2} . By writing so, we will assume that we are using the usual manifold inner products.

Recall that D maps onto the domain, i.e.:

$$D : L^2(S_1) \to L^2(S_1)$$

Then, two shape differences V_{S_1,S_2} and V_{S_1,S_3} are still functions on S_1 even if they are defined using maps to different shapes.

This allows us to "compare differences" (more on this tomorrow).

Discretization: standard basis

$$h_a^2(F(f), F(g)) = h_a^1(f, D(g))$$

Applying the discretizations we are familiar with, we get:

$$f^T F^T M_2 F g = f^T M_1 D g$$

from which we obtain an explicit expression for *D*:

$$D = M_1^{-1} F^T M_2 F$$

doesn't scale well with the size of the shapes!

where we discretized the functional map **using the standard "hat" basis**.

Discretization: area ratios

$$D = M_1^{-1} F^T M_2 F$$

As a simple example, consider the case in which the two shapes have exactly the same tessellations, and the same vertex orderings. Then the functional map F is simply the identity matrix, and we have:

$$D = M_1^{-1} M_2$$

In this case we have the intuitive interpretation of the shape difference as the point-wise area ratios. Hence, D is capturing the **area distortion** induced by the correspondence.

It follows that if D is the identity, the underlying map is area-preserving.

Discretization: LB basis

$$h_a^2(F(f), F(g)) = h_a^1(f, D(g))$$

Using the Laplacian eigenbases, we get:

$$(\Phi_2 C \Phi_1^{-1} f)^T M_2 (\Phi_2 C \Phi_1^{-1} g) = (\Phi_1 \Phi_1^{-1} f)^T M_1 (\Phi_1 D \Phi_1^{-1} g)$$

$$\Downarrow$$

$$D = C^T C$$

From this expression, it follows the useful fact that **orthogonal *C* is associated to an area-preserving map**.

Conformal inner product

The formulation described in the previous slides does not depend from the specific choice of an inner product (the discretization does, though!).

Another commonly used inner product is the so-called **conformal inner product**, which is defined as the inner product of gradients:

$$h_c(f,g) = \int \langle \nabla f(x), \nabla g(x) \rangle da(x)$$

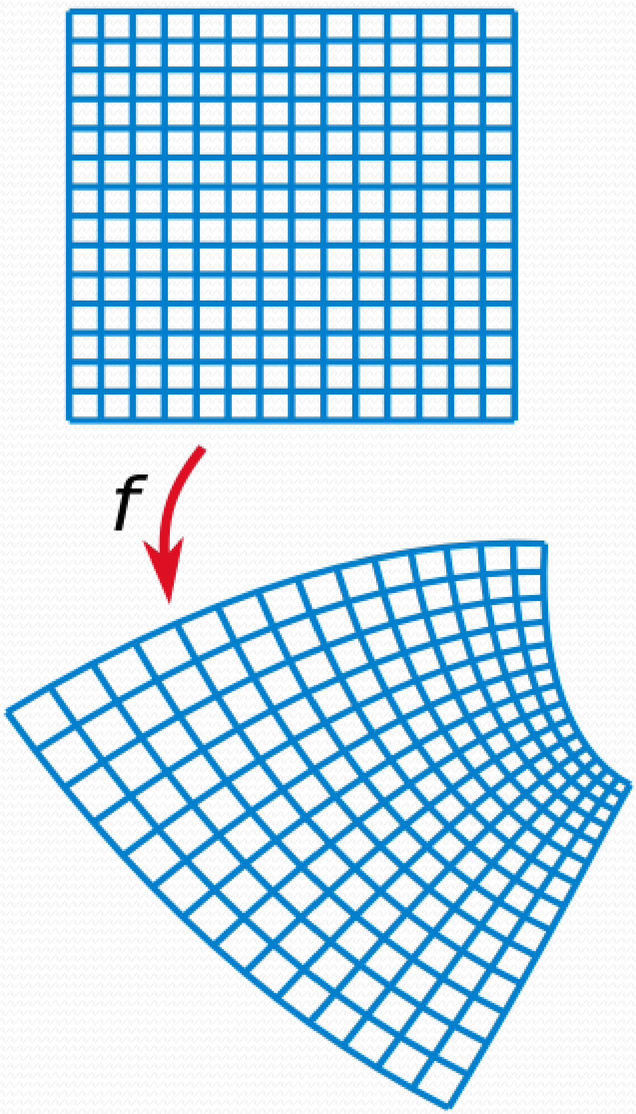
Similarly to the previous case, the following equality holds if and only if the **underlying map *T* is locally conformal**:

$$\boxed{h_c^1(f,g) = h_c^2(F(f), F(g))}$$

Conformal map

A conformal map is a function that **preserves angles** locally (have a look again at Lecture 6 – Slide 13).

Very common in computer graphics to work with conformal maps. This is mainly due to how texture mapping works.



Conformal-based difference

We can define a difference operator based on this inner product, just like before:

$$h_c^2(F(f), F(g)) = h_c^1(f, D(g))$$

To distinguish it from the previous area-based definition, we will rename this difference operator as R_{S_1,S_2}. Once again, this is an operator taking functions on S_1 and giving functions on S_1 itself.

To wrap it up, we now have two possible choices to define a shape difference, and these are given by choosing different inner products:

$$h_a^2(F(f), F(g)) = h_a^1(f, D(g))$$

$$h_c^2(F(f), F(g)) = h_c^1(f, D(g))$$

Discretization

$$h_c^2(F(f), F(g)) = h_c^1(f, D(g))$$

From the divergence theorem we know:

$$\int \langle \nabla f(x), \nabla g(x) \rangle da(x) = -\int f(x) \Delta g(x) da(x)$$

which means we can discretize the equality on top as:

$$-(Ff)^T M_2 M_2^{-1} W_2 F g = -f^T M_1 M_1^{-1} W_1 D g \Rightarrow D = W_1^{-1} F^T W_2 F$$

$$D = M_1^{-1} F^T M_2 F$$

compare with the previous case!

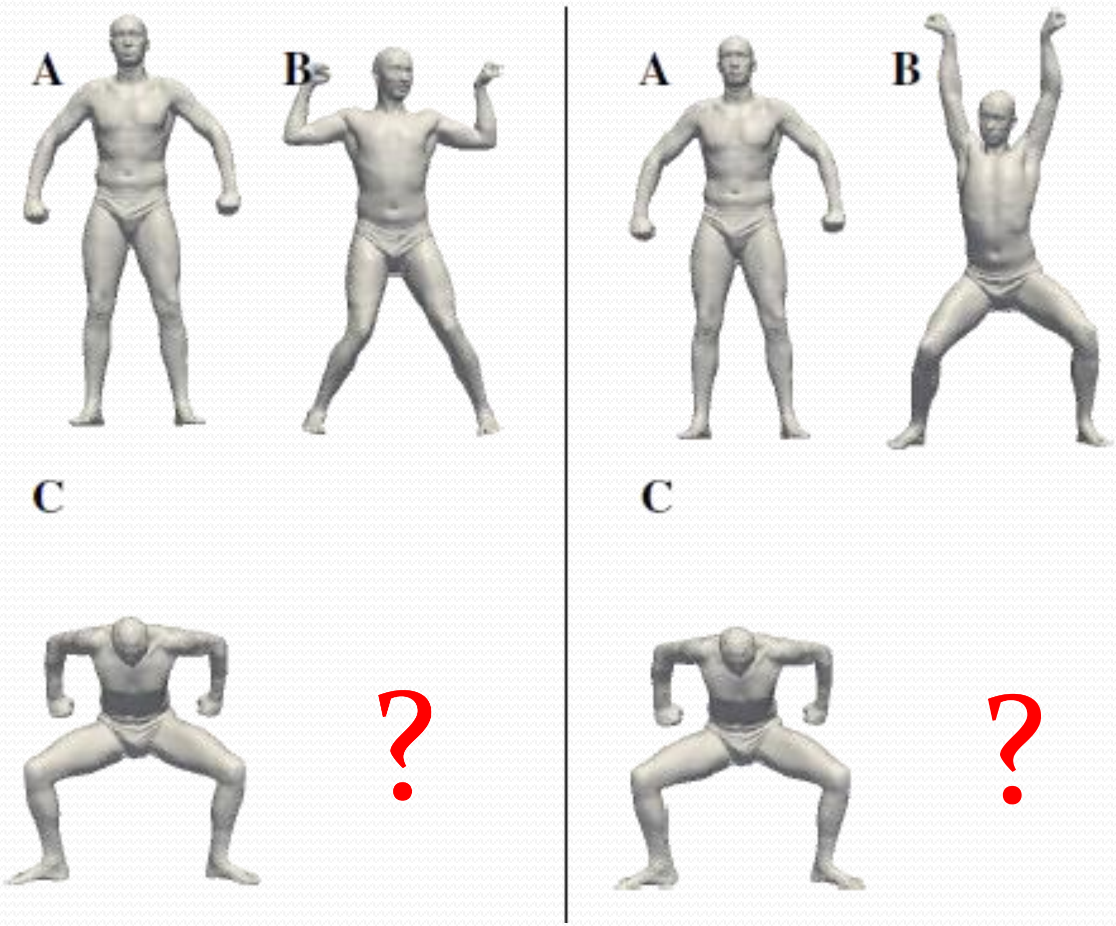
Similarly, using the LB basis we get:

$$D = \Lambda_1^{-1} C^T \Lambda_2 C$$

$$D = C^T C$$

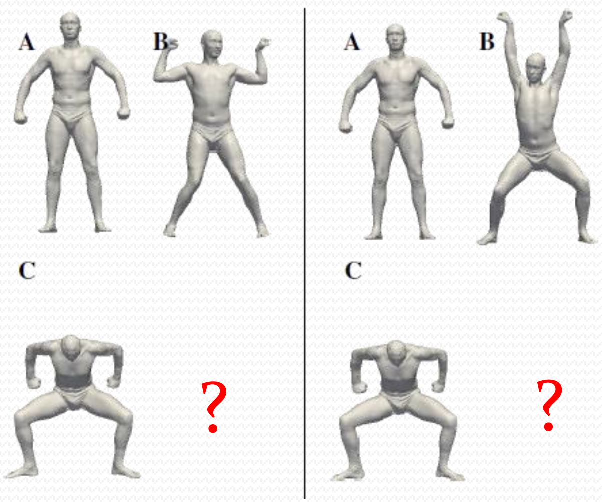
previous case

Shape analogies: problem



Shape analogies: solution

Assume we are given a collection of shapes, and we wish to find the shapes in the collection to fill up the question marks in the previous slide.



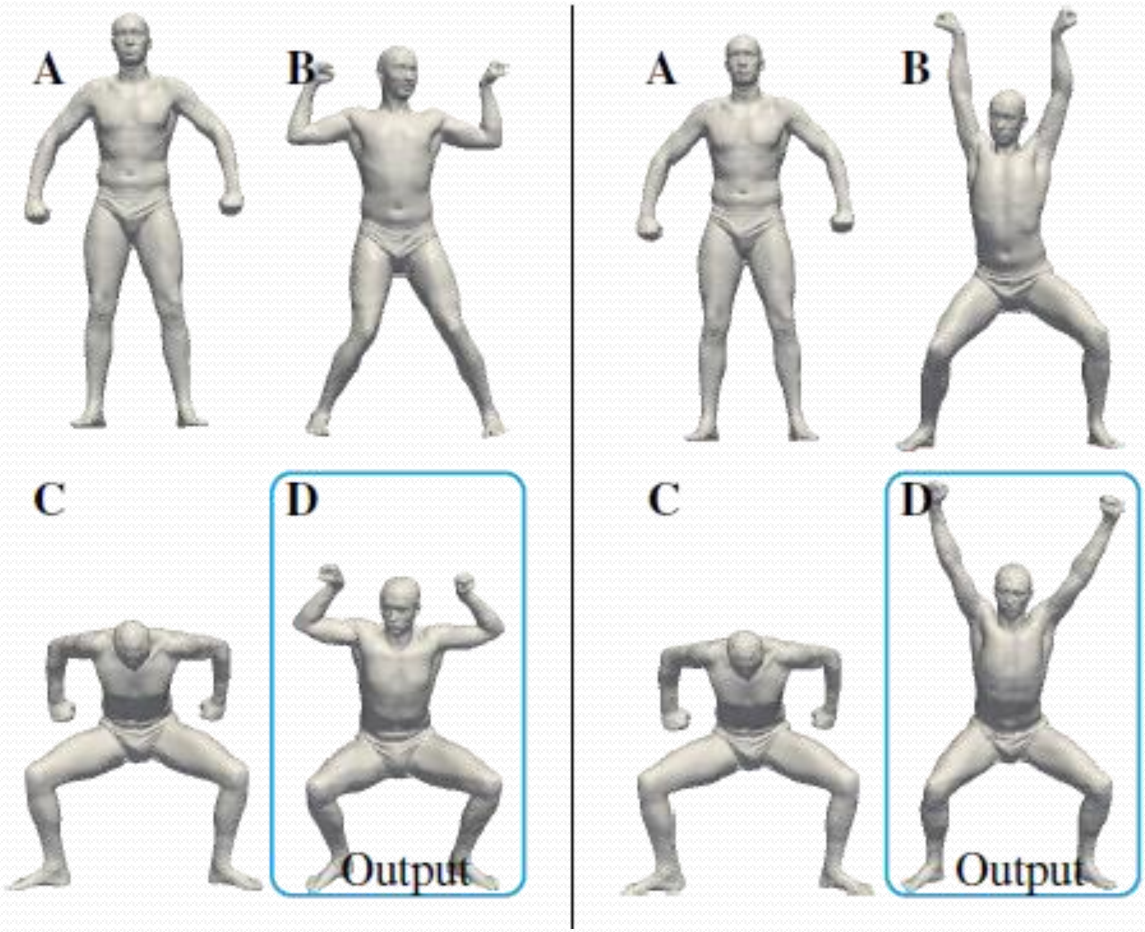
Suppose we have all the functional maps, including $G : L^2(A) \to L^2(C)$.

Then, we can exhaustively explore the whole shape collection, and retain the shape *S* such that:

$$S = \arg\min_X \|V_{C,X}G - GV_{A,B}\|_F^2 + \|R_{C,X}G - GR_{A,B}\|_F^2$$

G is used to transport the differences to a *common* comparison ground, namely the shape *C*.

Shape analogies: results



Suggested reading

- *Map-based exploration of intrinsic shape differences and variability*. Rustamov et al. TOG 2013.