

Weekly Exercises 2

Room: 02.09.023

Wed, 29.04.2015, 14:00-16:00

Submission deadline: Tue, 28.04.2015, 23:59 to windheus@in.tum.de

Mathematics: Metric Spaces

Exercise 1 (From the lecture, One point). *Show that...*

1. ... every isometry is an homeomorphism.
2. ... every surjective, distance-preserving map is injective (and thus an isometry).
3. ... being isometric is an equivalence relation.
4. ... $f : X \rightarrow \mathbb{R}, f(x) \mapsto \text{dist}_X(x, S)$ is a non-expanding function, where S is some fixed non-empty subset of X . Recall from the lecture that a function is non-expanding if it is Lipschitz with constant $C = 1$.

Let $f : X \rightarrow Y, g : Y \rightarrow Z$ be bi-Lipschitz homeomorphisms. *Show that...*

5. ... $g \circ f : X \rightarrow Z$ is a bi-Lipschitz homeomorphism.
6. ... $\text{dil}(g \circ f) \leq \text{dil}(f) \cdot \text{dil}(g)$
7. ... the Lipschitz distance is a metric.

Exercise 2 (Continuity, One point). Let $X \subset \mathbb{R}^2, Y \subset \mathbb{R}^3$. Find metrics d_X, d_Y , such that...

1. ... every function $f : X \rightarrow Y$ is continuous.
2. ... the only continuous functions $f : X \rightarrow Y$ are the constant functions.

Show that...

3. ... every Lipschitz function is continuous.
4. ... there exist continuous functions that are not Lipschitz.

Programming: Farthest Point Sampling

Exercise 3 (One point). Download and expand the file `exercise2.zip` from the lecture website. Modify the files `euclideanfps.m`, `euclideanvoronoi.m` to implement the functions as explained below. You can run the script `exercise.m` to test and visualize your solutions.

1. Given a triangle mesh $\mathcal{M} = (\mathcal{V}, \mathcal{F})$, a metric $d : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}_{\geq 0}$, a number $K \in \mathbb{N}$ and an initial vertex $v_1 \in \mathcal{V}$, the farthest-point-sampling method computes a set of K vertices by the following algorithm:

```
for  $i = 2$  to  $K$  do  
     $v_i \leftarrow \arg \max_{v \in \mathcal{V}} (\min_{j \in \{1, \dots, i-1\}} d(v, v_j))$   
end for  
return  $\{v_1, \dots, v_K\}$ 
```

Implement function `euclideanfps` that computes the farthest point sampling of a mesh with respect to the euclidean metric. The resulting set of vertices should be returned as a vector of indices.

2. Given a triangle mesh $\mathcal{M} = (\mathcal{V}, \mathcal{F})$ and a metric $d : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}_{\geq 0}$, the Voronoi cells with respect to a set of $K \in \mathbb{N}$ vertices $\{v_1, \dots, v_K\} \subset \mathcal{V}$ are defined as the partition $\mathcal{V}_1 \dot{\cup} \dots \dot{\cup} \mathcal{V}_K \dot{\cup} \mathcal{B} = \mathcal{V}$ such that

$$v \in \mathcal{V}_i \iff (\forall j \neq i : d(v, v_i) < d(v, v_j)),$$

for all $i \in \{1, \dots, K\}, v \in \mathcal{V}$.

If we assume $\mathcal{B} = \emptyset$ the partition into Voronoi cells can be represented by a function $f : \mathcal{V} \rightarrow \{1, \dots, K\}$. Implement function `euclideanvoronoi` that computes this partition of vertices into Voronoi cells with respect to the euclidean metric. (If you encounter vertices lying on the boundary \mathcal{B} , just pick the index of one of the neighbouring cells.)

Exercise 4 (One point). Modify the matlab files `metricfps.m`, `metricvoronoi.m`, `distortion.m` and `ghdistance.m` to implement the functions as explained below. You can run the script `exercise.m` to test and visualize your solutions.

1. Implement function `metricfps` that computes the farthest point sampling of a mesh with respect to a metric given by matrix $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{V}|}$. The resulting set of vertices should be returned as a vector of indices.
2. Implement function `metricvoronoi` that computes the partition of vertices into Voronoi cells with respect to a metric given by matrix $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{V}|}$.
3. Now we want to match two shapes, i.e. two set of vertices $\mathcal{V}_1, \mathcal{V}_2$, both equipped with a metric represented by $D_1 \in \mathbb{R}^{|\mathcal{V}_1| \times |\mathcal{V}_1|}, D_2 \in \mathbb{R}^{|\mathcal{V}_2| \times |\mathcal{V}_2|}$. To make the task computationally tractable we will use farthest point sampling to obtain two smaller subsets $S_1 \subset \mathcal{V}_1, S_2 \subset \mathcal{V}_2$, such that $|S_1| = |S_2|$. We will represent the metric D_1 restricted to S_1 by matrix $\hat{D}_1 \in \mathbb{R}^{|S_1| \times |S_1|}$. Analogically define

$\hat{D}_2 \in \mathbb{R}^{|S_2| \times |S_2|}$. A surjective correspondence can be represented by function $f : \{1, \dots, |S_1|\} \rightarrow \{1, \dots, |S_2|\}$, where $f(i) = j$ means that vertex $v_i \in S_1$ is set into correspondence with $v_j \in S_2$.

Implement function `distortion` that computes the distortion of given function $f : \{1, \dots, |S_1|\} \rightarrow \{1, \dots, |S_2|\}$ with respect to metrics $\hat{D}_1 \in \mathbb{R}^{|S_1| \times |S_1|}$, $\hat{D}_2 \in \mathbb{R}^{|S_2| \times |S_2|}$. Recall that the distortion is defined by

$$\text{dis}(f) = \sup_{v, w \in S_1} |\hat{D}_1(v, w) - \hat{D}_2(f(v), f(w))|.$$

4. Implement function `ghdistance` that computes the Gromov-Hausdorff distance between the two sets S_1, S_2 and also returns the optimal permutation p^* . You can use exhaustive search over the space of all permutations $p : \{1, \dots, |S_1|\} \rightarrow \{1, \dots, |S_2|\}$.