Analysis of Three-Dimensional Shapes
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Weekly Exercises 2

Room: 02.09.023

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Mathematics: Metric Spaces

Exercise 1 (From the lecture, One point). *Show that...*

1. *... every isometry is an homeomorphism.*
2. *... every surjective, distance-preserving map is injective (and thus an isometry).*
3. *... being isometric is an equivalence relation.*
4. *...* $f : X \to \mathbb{R}, f(x) \mapsto \mathrm{dist}_X(x, S)$ *is a nonexpanding function, where* S *is some fixed non-empty subset of* X.

Let $f : X \to Y$, $g : Y \to Z$ *be bi-Lipschitz homeomorphisms. Show that...*

5. *...* $g \circ f : X \to Z$ *is a bi-Lipschitz homeomorphism.*
6. *...* $\mathrm{dil}(g \circ f) \leq \mathrm{dil}(f) \cdot \mathrm{dil}(g)$
7. *... the Lipschitz distance is a metric.*

Exercise 2 (Continuity, One point). *Let* $X \subset \mathbb{R}^2, Y \subset \mathbb{R}^3$. *Find metrics* d_X, d_Y, *such that...*

1. *... every function* $f : X \to Y$ *is continuous.*
2. *... the only continuous functions* $f : X \to Y$ *are the constant functions.*

Show that...

3. *... every Lipschitz function is continuous.*
4. *... there exist continuous functions that are not Lipschitz.*

Programming: Farthest Point Sampling

Exercise 3 (One point). *Download and expand the file* `exercise2.zip` *from the lecture website. Modify the files* `euclideanfps.m`, `euclideanvoronoi.m` *to implement the functions as explained below. You can run the script* `exercise.m` *to test and visualize your solutions.*

1. *Given a triangle mesh* $\mathcal{M} = (\mathcal{V}, \mathcal{F})$*, a metric* $d : \mathcal{V} \times \mathcal{V} \to \mathbb{R}_{\geq 0}$*, a number* $K \in \mathbb{N}$ *and an initial vertex* $v_1 \in \mathcal{V}$*, the* farthest-point-sampling *method computes a set of* K *vertices by the following algorithm:*

   for $i = 2$ ***to*** K ***do***
   $v_i \leftarrow \arg\max_{v \in \mathcal{V}}(\min_{j \in \{1,\ldots,i-1\}} d(v, v_j))$
   end for
   return $\{v_1, \ldots, v_K\}$

   Implement function euclideanfps *that computes the farthest point sampling of a mesh with respect to the euclidean metric. The resulting set of vertices should be returned as a vector of indices.*

2. *Given a triangle mesh* $\mathcal{M} = (\mathcal{V}, \mathcal{F})$ *and a metric* $d : \mathcal{V} \times \mathcal{V} \to \mathbb{R}_{\geq 0}$*, the* Voronoi cells *with respect to a set of* $K \in \mathbb{N}$ *vertices* $\{v_1, \ldots, v_K\} \subset \mathcal{V}$ *are defined as the partition* $\mathcal{V}_1 \dot{\cup} \ldots \dot{\cup} \mathcal{V}_K \dot{\cup} \mathcal{B} = \mathcal{V}$ *such that*

$$v \in \mathcal{V}_i \iff (\forall j \neq i : d(v, v_i) < d(v, v_j)),$$

   for all $i \in \{1, \ldots, K\}, v \in \mathcal{V}$.

   If we assume $\mathcal{B} = \emptyset$ *the partition into Voronoi cells can be represented by a function* $f : \mathcal{V} \to \{1, \ldots, K\}$. *Implement function* euclideanvoronoi *that computes this partition of vertices into Voronoi cells with respect to the euclidean metric. (If you encounter vertices lying on the boundary* $\mathcal{B}$*, just pick the index of one of the neighbouring cells.)*

Exercise 4 (One point). *Modify the matlab files* `metricfps.m`, `metricvoronoi.m`, `distortion.m` *and* `ghdistance.m` *to implement the functions as explained below. You can run the script* `exercise.m` *to test and visualize your solutions.*

1. *Implement function* metricfps *that computes the farthest point sampling of a mesh with respect to a metric given by matrix* $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{V}|}$. *The resulting set of vertices should be returned as a vector of indices.*

2. *Implement function* metricvoronoi *that computes the partition of vertices into Voronoi cells with respect to a metric given by matrix* $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{V}|}$.

3. *Now we want to match two shapes, i.e. two set of vertices* $\mathcal{V}_1, \mathcal{V}_2$*, both equipped with a metric represented by* $D_1 \in \mathbb{R}^{|\mathcal{V}_1| \times |\mathcal{V}_1|}, D_2 \in \mathbb{R}^{|\mathcal{V}_2| \times |\mathcal{V}_2|}$. *To make the task computationally tractable we will use farthest point sampling to obtain two smaller subsets* $S_1 \subset \mathcal{V}_1, S_2 \subset \mathcal{V}_2$*, such that* $|S_1| = |S_2|$. *We will represent the metric* D_1 *restricted to* S_1 *by matrix* $\hat{D}_1 \in \mathbb{R}^{|S_1| \times |S_1|}$. *Analogically define*

$\hat{D}_2 \in \mathbb{R}^{|S_2|\times|S_2|}$. *A surjective correspondence can be represented by function* $f : \{1,\dots,|S_1|\} \to \{1,\dots,|S_2|\}$, *where* $f(i) = j$ *means that vertex* $v_i \in S_1$ *is set into correspondence with* $v_j \in S_2$.

Implement function distortion *that computes the distortion of given function* $f : \{1,\dots,|S_1|\} \to \{1,\dots,|S_2|\}$ *with respect to metrics* $\hat{D}_1 \in \mathbb{R}^{|S_1|\times|S_1|}, \hat{D}_2 \in \mathbb{R}^{|S_2|\times|S_2|}$. *Recall that the* distortion *is defined by*

$$dis(f) = \sup_{v,w\in S_1} |\hat{D}_1(v,w) - \hat{D}_2(f(v), f(w))|.$$

4. *Implement function* ghdistance *that computes the Gromov-Hausdorff distance between the two sets* S_1, S_2 *and also returns the optimal permutation* p^*. *You can use exhaustive search over the space of all permutations* $p : \{1,\dots,|S_1|\} \to \{1,\dots,|S_2|\}$.