Analysis of Three-Dimensional Shapes
E. Rodolà, T. Windheuser, M. Vestner
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Computer Vision Group
Institut für Informatik
Technische Universität München

Weekly Exercises 3

Room: 02.09.023

Wed, 06.05.2015, 14:15-15:45

Submission deadline: Wed, 06.05.2015, 11:59 am to windheus@in.tum.de

Please send in only Latex-PDF. If you have hand-written solutions, please hand them in during the lecture.

Mathematics: Curves in Space

Definition (Curves)**.** *Let $I \subset \mathbb{R}$ be an open interval of $\mathbb{R}$. A* curve *is a smooth map $\alpha : I \to \mathbb{R}^3$. The curve is* regular*, if $\alpha'(s) \neq \mathbf{0}$ for any $s \in I$. The derivative $\alpha'(s)$ with respect to s is called the* tangent vector *at s or at point $\alpha(s)$.*

Definition (Length of Curves)**.** *Let $[a, b] \subset I$ be a compact interval and $\alpha : I \to \mathbb{R}^3$ be a regular curve. The* length $L_a^b(\alpha)$ of α from a to b *is defined by*

$$L_a^b(\alpha) = \int_a^b \|\alpha'(s)\|\, ds = \int_a^b \sqrt{\langle \alpha'(s), \alpha'(s) \rangle} ds, \tag{1}$$

where $\|\cdot\|$ is the usual ℓ_2-norm and $\langle \cdot, \cdot \rangle$ the standard inner product of $\mathbb{R}^3$.

Definition (Rigid Motion)**.** *A transformation $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is called a* rigid motion *if there exists a matrix $A \in O(n)$ and a vector $b \in \mathbb{R}^n$, such that $\phi(x) = Ax + b$ for all $x \in \mathbb{R}^n$.*

Recall that $A \in O(n) \Leftrightarrow AA^\top = \mathrm{Id} \Leftrightarrow \det A = \pm 1 \Leftrightarrow \phi$ is an isometry of $\mathbb{R}^n$. Note that by this definition reflections ($\det A = -1$) are also rigid motions. Other literature might exclude reflections from the set of rigid motions.

Exercise 1 (One point)**.**

1. *Let $\alpha : (-4, 4) \to \mathbb{R}^3, s \mapsto (s, 2s, s^2 + 1)$ be a curve. Show that the curve is regular and compute the tangent vector of α at $s \in (-4, 4)$ and the length $L_{-2}^2(\alpha)$.*
2. *Let $\alpha : (-4, 4) \to \mathbb{R}^3, s \mapsto (s, 2s, s^2+1)$ be a regular curve and let $\gamma : (-2, 2) \to (-4, 4), s \mapsto 2s$ be a function. Show that the curve $\beta = \alpha \circ \gamma$ is regular and compute the tangent vector of β at $s \in (-2, 2)$ and the length $L_{-1}^1(\beta)$.*
3. *Let $\alpha : I \to \mathbb{R}^3$ be a regular curve, $[a, b] \subset I$ and let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be a rigid motion. Show that $L_a^b(\phi \circ \alpha) = L_a^b(\alpha)$.*

Mathematics: Surfaces in Space

Definition (Surface)**.** *A non-empty set* $\mathcal{X} \subset \mathbb{R}^3$ *is called a* surface *if, for each* $p \in \mathcal{X}$*, there exists an open neighbourhood* $N \subset \mathbb{R}^3$*, an open set* $U \subset \mathbb{R}^2$ *and a differentiable map* $x : U \to V$*, where* $V = \mathcal{X} \cap N$*, such that it holds:*

1. $x : U \to V$ *is a homeomorphism and*
2. *the partial derivatives* $x_u(q) = \frac{\partial x}{\partial u}(q) \in \mathbb{R}^3$ *and* $x_v(q) = \frac{\partial x}{\partial v}(q) \in \mathbb{R}^3$ *are non-zero and linearly independent for all* $q \in U \subset \mathbb{R}^2$*.*

Recall that a homeomorphism $x : U \to V$ is a continuous and bijective map, such that the inverse x^{-1} is also continuous. The set $\mathcal{T}_p(\mathcal{X}) = \operatorname{span}(x_u(x^{-1}(p)), x_v(x^{-1}(p)))$ is a 2-dimensional subspace of $\mathbb{R}^3$ and is called the *tangent space* of $\mathcal{X}$ at point p.

The individual maps x are called *charts* or *parameterizations* and a collection of charts covering $\mathcal{X}$ is sometimes called an *atlas*.

Exercise 2 (One point)**.** *Let* $U = (-4, 4) \times (-4, 4) \subset \mathbb{R}^2$ *and let* $x : U \to \mathbb{R}^3, \binom{u}{v} \mapsto (u, v, (u + v)^2 + 1)^\top$ *be a chart of surface* $\mathcal{X} = Im(x)$*.*

1. *Compute the partial derivatives* $x_u\binom{u}{v} = \frac{\partial x}{\partial u}\binom{u}{v}, x_v\binom{u}{v} = \frac{\partial x}{\partial v}\binom{u}{v}$ *for any* $\binom{u}{v} \in U$*.*
2. *Compute the differential* $(dx)_{\binom{0.5}{1}}\binom{2}{1}$ *of* x *at point* $\binom{0.5}{1} \in U$ *in direction* $\binom{2}{1}$*.*