

Weekly Exercises 3

Room: 02.09.023

Wed, 06.05.2015, 14:15-15:45

Submission deadline: Wed, 06.05.2015, 11:59 am to windheus@in.tum.de

Please send in only Latex-PDF. If you have hand-written solutions, please hand them in during the lecture.

Mathematics: Curves in Space

Definition (Curves). Let $I \subset \mathbb{R}$ be an open interval of \mathbb{R} . A curve is a smooth map $\alpha : I \rightarrow \mathbb{R}^3$. The curve is regular, if $\alpha'(s) \neq \mathbf{0}$ for any $s \in I$. The derivative $\alpha'(s)$ with respect to s is called the tangent vector at s or at point $\alpha(s)$.

Definition (Length of Curves). Let $[a, b] \subset I$ be a compact interval and $\alpha : I \rightarrow \mathbb{R}^3$ be a regular curve. The length $L_a^b(\alpha)$ of α from a to b is defined by

$$L_a^b(\alpha) = \int_a^b \|\alpha'(s)\| ds = \int_a^b \sqrt{\langle \alpha'(s), \alpha'(s) \rangle} ds, \quad (1)$$

where $\|\cdot\|$ is the usual ℓ_2 -norm and $\langle \cdot, \cdot \rangle$ the standard inner product of \mathbb{R}^3 .

Definition (Rigid Motion). A transformation $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called a rigid motion if there exists a matrix $A \in O(n)$ and a vector $b \in \mathbb{R}^n$, such that $\phi(x) = Ax + b$ for all $x \in \mathbb{R}^n$.

Recall that $A \in O(n) \Leftrightarrow AA^\top = \text{Id} \Leftrightarrow \det A = \pm 1 \Leftrightarrow \phi$ is an isometry of \mathbb{R}^n . Note that by this definition reflections ($\det A = -1$) are also rigid motions. Other literature might exclude reflections from the set of rigid motions.

Exercise 1 (One point). 1. Let $\alpha : (-4, 4) \rightarrow \mathbb{R}^3, \alpha(s) \mapsto (s, 2s, s^2 + 1)$ be a curve. Show that the curve is regular and compute the tangent vector of α at $s \in (-4, 4)$ and the length $L_{-2}^2(\alpha)$.

2. Let $\alpha : (-4, 4) \rightarrow \mathbb{R}^3, \alpha(s) \mapsto (s, 2s, s^2 + 1)$ be a regular curve and let $\gamma : (-2, 2) \rightarrow (-4, 4), \gamma(s) \mapsto 2s$ be a function. Show that the curve $\beta = \alpha \circ \gamma$ is regular and compute the tangent vector of β at $s \in (-2, 2)$ and the length $L_{-1}^1(\beta)$.

3. Let $\alpha : I \rightarrow \mathbb{R}^3$ be a regular curve, $[a, b] \subset I$ and let $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a rigid motion. Show that $L_a^b(\phi \circ \alpha) = L_a^b(\alpha)$.

Mathematics: Surfaces in Space

Definition (Surface). A non-empty set $\mathcal{X} \subset \mathbb{R}^3$ is called a surface if, for each $p \in \mathcal{X}$, there exists an open set $U \subset \mathbb{R}^2$ and a differentiable map $x : U \rightarrow \mathcal{X}$, such that it holds:

1. $p \in V$, where $V = \text{Im}(x) \subset \mathcal{X}$,
2. $x : U \rightarrow V$ is a homeomorphism and
3. the partial derivatives $x_u(q) = \frac{\partial x}{\partial u}(q) \in \mathbb{R}^3$ and $x_v(q) = \frac{\partial x}{\partial v}(q) \in \mathbb{R}^3$ are non-zero and linearly independent for all $q \in U \subset \mathbb{R}^2$.

Recall that a homeomorphism $x : U \rightarrow V$ is a continuous and bijective map, such that the inverse x^{-1} is also continuous. The set $\mathcal{T}_p(\mathcal{X}) = \text{span}(x_u(x^{-1}(p)), x_v(x^{-1}(p)))$ is a 2-dimensional subspace of \mathbb{R}^3 and is called the *tangent space* of \mathcal{X} at point p .

The individual maps x are called *charts* and a collection of charts covering \mathcal{X} is called a *parameterization* or sometimes an *atlas*.

Exercise 2 (One point). Let $U = (-4, 4) \times (-4, 4) \subset \mathbb{R}^2$ and let $x : U \rightarrow \mathbb{R}^3, x\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) \mapsto (u, v, (u+v)^2 + 1)^\top$ be a chart of surface $\mathcal{X} = \text{Im}(x)$. Let $\alpha : (-2, 2) \rightarrow U, \alpha(s) \mapsto (s, 2s)^\top$ be a regular curve.

1. Compute the partial derivatives $x_u\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) = \frac{\partial x}{\partial u}\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right), x_v\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) = \frac{\partial x}{\partial v}\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)$ for any $\begin{pmatrix} u \\ v \end{pmatrix} \in U$.
2. Compute the differential $(dx)_{\begin{pmatrix} 0.5 \\ 1 \end{pmatrix}}\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$ of x at point $\begin{pmatrix} 0.5 \\ 1 \end{pmatrix} \in U$ in direction $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.