

Weekly Exercises 4

Room: 02.09.023

Wed, 13.05.2015, 14:15-15:45

Submission deadline: Tue, 12.05.2015, 23:59 to windheus@in.tum.de

Please send in only Latex-PDF. If you have hand-written solutions, please hand them in during the lecture.

Surfaces in Space

Definition (Surface). A non-empty set $\mathcal{X} \subset \mathbb{R}^3$ is called a surface if, for each $p \in \mathcal{X}$, there exists an open neighbourhood $N \subset \mathbb{R}^3$, an open set $U \subset \mathbb{R}^2$ and a differentiable map $x : U \rightarrow V$, where $V = \mathcal{X} \cap N$, such that it holds:

1. $x : U \rightarrow V$ is a homeomorphism and
2. the partial derivatives $x_u(q) = \frac{\partial x}{\partial u}(q) \in \mathbb{R}^3$ and $x_v(q) = \frac{\partial x}{\partial v}(q) \in \mathbb{R}^3$ are non-zero and linearly independent for all $q \in U \subset \mathbb{R}^2$.

Recall that a homeomorphism $x : U \rightarrow V$ is a continuous and bijective map, such that the inverse x^{-1} is also continuous. The set $\mathcal{T}_p(\mathcal{X}) = \text{span}(x_u(x^{-1}(p)), x_v(x^{-1}(p)))$ is a 2-dimensional subspace of \mathbb{R}^3 and is called the *tangent space* of \mathcal{X} at point p .

The individual maps x are called *charts* or *parameterizations* and a collection of charts covering \mathcal{X} is sometimes called an *atlas*.

Exercise 1 (One Point). Let \mathcal{X} be a surface and let $x : U_x \rightarrow V \subset \mathcal{X}$ be a chart of \mathcal{X} .

1. Let $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \phi(q) \mapsto Aq + b$ an affine transformation, such that $A \in \mathbb{R}^2, b \in \mathbb{R}^2, \det(A) \neq 0$. Let $U_y = \phi^{-1}(U_x)$ and define $y : U_y \rightarrow \mathcal{X}, q \mapsto x \circ \phi(q)$. Show that for any $p \in V$ it holds

$$\text{span}(x_u(x^{-1}(p)), x_v(x^{-1}(p))) = \text{span}(y_u(y^{-1}(p)), y_v(y^{-1}(p))).$$

2. Let $y : U_y \rightarrow V \subset \mathcal{X}$ be a chart and $\phi : U_x \rightarrow U_y$ be the homeomorphism between U_x and U_y defined by $\phi = y^{-1} \circ x$. Show that if we assume ϕ is differentiable it holds for any $p \in V$

$$\text{span}(x_u(x^{-1}(p)), x_v(x^{-1}(p))) = \text{span}(y_u(y^{-1}(p)), y_v(y^{-1}(p))).$$

Remark. Exercise 1 shows that if the composition $y^{-1} \circ x$ of charts x, y is differentiable the tangent plane is well defined, i.e. it does not depend on the parameterization. The proposition mentioned in the lecture tells us, that this is the case for all charts of a surface:

Proposition. Let $x : U_x \rightarrow V \subset \mathcal{X}, y : U_y \rightarrow V \subset \mathcal{X}$ be two charts of surface \mathcal{X} . The composition $y^{-1} \circ x$ is a differentiable function.

The proof is somewhat technical but can be found in any textbook on differential geometry. Another important observation is that $\mathcal{T}_{x(q)}(\mathcal{X}) = \text{span}(x_u(q), x_v(q)) = \text{Im}(dx)_{x(q)}$, i.e. $dx : \mathbb{R}^2 \rightarrow \mathcal{T}_{x(q)}(\mathcal{X})$ is a linear bijective map between \mathbb{R}^2 and the tangent plane $\mathcal{T}_{x(q)}(\mathcal{X})$.

Definition (Integral on Surfaces). Let \mathcal{X} be a surface, let $x : U \rightarrow V \subset \mathcal{X}$ be a chart and let $f : \mathcal{X} \rightarrow \mathbb{R}$ be a real valued function. We define the integral of f on subsurface V with respect to x by

$$\int_V f dp = \int_U f(x(q)) \sqrt{\det((dx)_q^\top (dx)_q)} dq.$$

The area $\mathcal{A}(V)$ of V is defined by setting $f(p) = 1$ everywhere, i.e. $\mathcal{A}(V) = \int_V 1 dp$.

Definition (Differential of Functions on Surfaces). Let \mathcal{X} be a surface. The differential of some vector-valued function $f : \mathcal{X} \rightarrow \mathbb{R}^m$ at point $p \in \mathcal{X}$ is a function $(df)_p : \mathcal{T}_p(\mathcal{X}) \rightarrow \mathbb{R}^m$ defined by

$$(df)_p(w) = (f \circ \alpha)'(s_0), \tag{1}$$

where $\alpha : (s_0 - \epsilon, s_0 + \epsilon) \rightarrow \mathcal{X}, s_0 \in \mathbb{R}$ is any regular curve, such that $\alpha(s_0) = p$ and $\alpha'(s_0) = w$.

Loosely speaking, the differential $(df)_p(w)$ is the derivative of f when moving on the surface at point p in direction w . The domain of $(df)_p$ is $\mathcal{T}_p(\mathcal{X})$ since this is the space of all possible directions at p .

Exercise 2 (One Point). 1. Show that the integral over $V \subset \mathcal{X}$ is well defined, i.e. it does not depend on the choice of the chart.

2. Show that the differential of some function f on a surface is well defined, i.e. it does not depend on the choice of the curve.

3. Show that $(df)_p : \mathcal{T}_p(\mathcal{X}) \rightarrow \mathbb{R}^m$ is a linear map.

Hint. Recall that the differential df of some function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ between real-valued vector spaces is well defined and the chain rule $(d(f \circ g))_a = (df)_{g(a)}(dg)_a$ can be applied. For the second part of the exercise proving $(f \circ \alpha)'(0) = (d(f \circ x))_{x^{-1}(p)}(dx^{-1})_p w$ might be helpful.

Exercise 3 (One Point). 1. Compute the following integral:

$$\int_0^1 \int_0^{1-u} a(1-u-v) + bu + cv \, dvdu.$$

2. Let $a, b, c \in \mathbb{R}^3$ be the corners of some triangle T , such that $T \subset \mathcal{X}$ is part of surface \mathcal{X} . Consider two functions $f, g : \mathcal{X} \rightarrow \mathbb{R}$ that are linear on T and take values $f(a), g(b) \in \mathbb{R}, f(b) = f(c) = g(a) = g(c) = 0$ at it's corners. Compute the integral on the interior $\text{int}(T)$ of T :

$$\int_{\text{int}(T)} f(t)g(t)dt.$$