Computer Vision Group
Institut für Informatik
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Weekly Exercises 5

Room: 02.09.023

Wed, 27.05.2015, 14:15-15:45

Submission deadline: Tue, 26.05.2015, 23:59 to windheus@in.tum.de

Please send in only Latex-PDF. If you have hand-written solutions, please hand them in during the lecture.

Discrete Laplace Beltrami Operator

Exercise 1 (One point). *Let $\mathcal{F}(\mathcal{X}, \mathbb{R})$ be a vector space of real valued functions on a surface $\mathcal{X}$ spanned by $\Phi = \{\phi_i\}_{i=1\dots n}$. Define the inner product $\langle \cdot, \cdot \rangle$ between two functions $f, g \in \mathcal{F}(\mathcal{X}, \mathbb{R})$ by*

$$\langle f, g \rangle = \int_{\mathcal{X}} f(p)g(p)dp.$$

1. *Show that $\langle \cdot, \cdot \rangle$ is a bilinear map that can be represented by a matrix $M \in \mathbb{R}^{n \times n}$, such that $\langle f, g \rangle = \alpha^\top M \beta$ for any functions $f, g \in \mathcal{F}(\mathcal{X}, \mathbb{R})$ with coordinates $\alpha, \beta \in \mathbb{R}^n$ with respect to basis Φ.*

2. *Let now $\mathcal{X} = (\mathcal{V} = \{v_1, \dots, v_n\}, T = \{t_1, \dots, t_m\})$ be a discrete triangular mesh and let the basis functions $\phi_i : \mathcal{X} \to \mathbb{R}$ behave linearly inside each traingle and satisfy*

$$\phi_i(v_i) = 1,$$
$$\phi_i(v_j) = 0, \; \forall j \neq i$$

on the vertices.

Calculate

$$M^k_{i.j} = \int_{t_k} \phi_i(p)\phi_j(p)dp$$

for each triangle $t_k \in T$. What is the relation between $M^k_{i.j}$ and $M_{i.j}$?

Exercise 2 (One point). *The Laplace operator on a regular surface $\mathcal{X}$ (without boundary) is a linear operator $\Delta : \mathcal{F}_1(\mathcal{X}, \mathbb{R}) \to \mathcal{F}_2(\mathcal{X}, \mathbb{R})$ between two appropriate function spaces, such that for all $g \in \mathcal{F}_3(\mathcal{X}, \mathbb{R})$ from a test-space it holds*

$$\int_{\mathcal{X}} (\Delta f)(p)g(p)dp = -\int_{\mathcal{X}} \langle \nabla f(p), \nabla g(p) \rangle dp. \tag{1}$$

Let now $\mathcal{X} = (\mathcal{V} = \{v_1, \dots, v_n\}, T = \{t_1, \dots, t_m\})$ be a discrete triangular mesh and let the function spaces $\mathcal{F}_1(\mathcal{X}, \mathbb{R}) = \mathcal{F}_2(\mathcal{X}, \mathbb{R}) = \mathcal{F}_3(\mathcal{X}, \mathbb{R}) = span\{\phi_i | i = 1 \dots, n\}$ be spanned by the basis functions $\phi_i : \mathcal{X} \to \mathbb{R}$ as defined in the first exercise.

1. *Show that* $\int_{\mathcal{X}} \langle \nabla f(p), \nabla g(p) \rangle dp$ *is a bilinear map that can be represented by a matrix* $S \in \mathbb{R}^{n \times n}$*, such that* $\int_{\mathcal{X}} \langle \nabla f(p), \nabla g(p) \rangle dp = \alpha^\top S \beta$ *for any functions* $f, g \in \mathcal{F}(\mathcal{X}, \mathbb{R})$ *with coordinates* $\alpha, \beta \in \mathbb{R}^n$ *with respect to basis* Φ*.*

2. *Calculate the integrals*

$$S_{ij}^k = \int_{t_k} \langle \nabla \phi_i(x), \nabla \phi_j(x) \rangle dx$$

for each triangle $t_k \in T$*. What is the relation between* $S_{i.j}^k$ *and* $S_{i.j}$*?*

3. *The linear operator* Δ *becomes a matrix* $L \in \mathbb{R}^{n \times n}$*, such that for a given function* $f(x) = \sum_{i=1}^n \alpha_i \phi_i(x)$ *it holds* $\Delta f(x) = h(x) = \sum_{j=1}^n \gamma_j \phi_j(x)$ *with* $\gamma = L\alpha$*. Find an expression for* L *in terms of* M *and* S*.*

Programming: Multi-Dimensional Scaling

Exercise 3 (Two points)**.** *Download and expand the file* `exercise4.zip` *from the lecture website. Modifiy the files* `mds.m` *and* `alignpoints.m` *to implement the functions as explained below. You can run the script* `exercise.m` *to test and visualize your solutions.*

`mds.m` *Implement the* multi-dimensional scaling *method introduced in the lecture. The function accepts a metric given by matrix* $D \in \mathbb{R}^{n \times n}$ *and a dimension* $m \in \mathbb{N}$*. It should return the coordinates of each of the* n *points embedded into* $\mathbb{R}^m$ *as matrix* $Z \in \mathbb{R}^{n \times m}$*.*

Parameters `alpha`*,* `epsilon` *and* `maxI` *control the gradient descent's behaviour and are the stepsize, minimum relative progress and maximum number of iterations, respectively.*

`alignpoints.m` *Write a function that aligns two point clouds. Given two set of points* $Z_1, Z_2 \subset \mathbb{R}^m$ *embedded into* $\mathbb{R}^m$ *by the MDS method find two rigid transformations that are represented by* $(R_1, \mathbf{t}_1), (R_2, \mathbf{t}_2) \in \mathbb{R}^{m \times m} \times \mathbb{R}^m$*. The transformed point sets* $\hat{Z}_1 = \{R_1(\mathbf{z} + \mathbf{t}_1) | \mathbf{z} \in Z_1\}, \hat{Z}_2 = \{R_2(\mathbf{z} + \mathbf{t}_2) | \mathbf{z} \in Z_2\}$ *should be aligned to each other.*

Translations $\mathbf{t}_1, \mathbf{t}_2$ *can be found by computing the point clouds' mean. There are several ways to find good rotation matrices. We suggest to align the principal axis* $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \in \mathbb{R}^m$ *of each point cloud with the standard Euclidean axis* $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m \in \mathbb{R}^m$*. Note that* $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ *form an orthonormal basis of* $\mathbb{R}^m$ *and can be computed by the matlab function* `pca`*.*

The matlab function `alignpoints` *should accept two point clouds by arguments* `Z1` *and* `Z2` *and should return the transformed poitns as* `Zhat1` *and* `Zhat2`*. The rotations and translations are not needed as return values.*

Note that since the signs of the principal axis are not uniquely determined, the visualization code in `exercise.m` *includes sign parameters that can be altered.*