Analysis of Three-Dimensional Shapes
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# Weekly Exercises 4 

Room: 02.09.023
Wed, 13.05.2015, 14:15-15:45
Submission deadline: Tue, 12.05.2015, 23:59 to windheus@in.tum.de Please send in only Latex-PDF. If you have hand-written solutions, please hand them in during the lecture.

## Surfaces in Space

Definition (Surface). A non-empty set $\mathcal{X} \subset \mathbb{R}^{3}$ is called a surface if, for each $p \in \mathcal{X}$, there exists an open neighbourhood $N \subset \mathbb{R}^{3}$, an open set $U \subset \mathbb{R}^{2}$ and a differentiable map $x: U \rightarrow V$, where $V=\mathcal{X} \cap N$, such that it holds:

1. $x: U \rightarrow V$ is a homeomorphism and
2. the partial derivatives $x_{u}(q)=\frac{\partial x}{\partial u}(q) \in \mathbb{R}^{3}$ and $x_{v}(q)=\frac{\partial x}{\partial v}(q) \in \mathbb{R}^{3}$ are nonzero and linearly independent for all $q \in U \subset \mathbb{R}^{2}$.

Recall that a homeomorphism $x: U \rightarrow V$ is a continuous and bijective map, such that the inverse $x^{-1}$ is also continuous. The set $\mathcal{T}_{p}(\mathcal{X})=\operatorname{span}\left(x_{u}\left(x^{-1}(p)\right), x_{v}\left(x^{-1}(p)\right)\right)$ is a 2-dimensional subspace of $\mathbb{R}^{3}$ and is called the tangent space of $\mathcal{X}$ at point $p$.

The individual maps $x$ are called charts or parameterizations and a collection of charts covering $\mathcal{X}$ is sometimes called an atlas.

Exercise 1 (One Point). Let $\mathcal{X}$ be a surface and let $x: U_{x} \rightarrow V \subset \mathcal{X}$ be a chart of $\mathcal{X}$.

1. Let $\phi: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}, \phi(q) \mapsto A q+b$ an affine transformation, such that $A \in \mathbb{R}^{2}, b \in$ $\mathbb{R}^{2}, \operatorname{det}(A) \neq 0$. Let $U_{y}=\phi^{-1}\left(U_{x}\right)$ and define $y: U_{y} \rightarrow \mathcal{X}, q \mapsto x \circ \phi(q)$. Show that for any $p \in V$ it holds

$$
\operatorname{span}\left(x_{u}\left(x^{-1}(p)\right), x_{v}\left(x^{-1}(p)\right)\right)=\operatorname{span}\left(y_{u}\left(y^{-1}(p)\right), y_{v}\left(y^{-1}(p)\right)\right) .
$$

2. Let $y: U_{y} \rightarrow V \subset \mathcal{X}$ be a chart and $\phi: U_{x} \rightarrow U_{y}$ be the homeomorphism between $U_{x}$ and $U_{y}$ defined by $\phi=y^{-1} \circ x$. Show that if we assume $\phi$ is differentiable it holds for any $p \in V$

$$
\operatorname{span}\left(x_{u}\left(x^{-1}(p)\right), x_{v}\left(x^{-1}(p)\right)\right)=\operatorname{span}\left(y_{u}\left(y^{-1}(p)\right), y_{v}\left(y^{-1}(p)\right)\right) .
$$

Solution. 1. Let $p \in V, q \in U_{y}$, such that $p=y(q)$. We can expand $y_{u}(q)$ to

$$
y_{u}(q)=\frac{\partial y}{\partial u}(q)=\frac{\partial x \circ \phi}{\partial u}(q)=\mathcal{J}_{\phi(q)}(x) \phi_{u}(q)=\left(x_{u}(\phi(q)), x_{v}(\phi(q))\right) \phi_{u}(q)
$$

Analoguously it holds

$$
y_{v}(q)=\left(x_{u}(\phi(q)), x_{v}(\phi(q))\right) \phi_{v}(q)
$$

So clearly linear combination $a y_{u}(q)+b y_{v}(q) \in \operatorname{span}\left(x_{u}(\phi(q)), x_{v}(\phi(q)), a, b \in\right.$ $\mathbb{R}$. By $\phi(q)=x^{-1} \circ y(q)=x^{-1}(p)$ it holds

$$
a y_{u}(q)+b y_{v}(q) \in \operatorname{span}\left(x_{u}\left(x^{-1}(p)\right), x_{v}\left(x^{-1}(p)\right) .\right.
$$

By the definition of a chart the pairs of derivatives $x_{u}, x_{v}$ and $y_{u}, y_{u}$ are linear independent, thus the proposition holds.
2. The proof is very similar to the first. Let $p \in V, q \in U_{x}$, such that $p=x(q)$. Transform $\phi=y^{-1} \circ x$ into $x=y \circ \phi$ and then expand $x_{u}(q)$ to

$$
x_{u}(q)=\frac{\partial x}{\partial u}(q)=\frac{\partial y \circ \phi}{\partial u}(q)=\mathcal{J}_{\phi(q)}(y) \phi_{u}(q)=\left(y_{u}(\phi(q)), y_{v}(\phi(q))\right) \phi_{u}(q) .
$$

Analoguously it holds

$$
x_{v}(q)=\left(y_{u}(\phi(q)), y_{v}(\phi(q))\right) \phi_{v}(q)
$$

So clearly linear combination $a x_{u}(q)+b x_{v}(q) \in \operatorname{span}\left(y_{u}(\phi(q)), y_{v}(\phi(q)), a, b \in\right.$ $\mathbb{R}$. By $\phi(q)=y^{-1} \circ x(q)=y^{-1}(p)$ it holds

$$
a x_{u}(q)+b x_{v}(q) \in \operatorname{span}\left(y_{u}\left(y^{-1}(p)\right), y_{v}\left(y^{-1}(p)\right) .\right.
$$

By the definition of a chart the pairs of derivatives $x_{u}, x_{v}$ and $y_{u}, y_{u}$ are linear independent, thus the proposition holds.

Remark. Exercise 1 shows that if the composition $y^{-1} \circ x$ of charts $x, y$ is differentiable the tangent plane is well defined, i.e. it does not depend on the parameterization. The proposition mentioned in the lecture tells us, that this is the case for all charts of a surface:

Proposition. Let $x: U_{x} \rightarrow V \subset \mathcal{X}, y: U_{y} \rightarrow V \subset \mathcal{X}$ be two charts of surface $\mathcal{X}$. The composition $y^{-1} \circ x$ is a differentiable function.

The proof is somewhat technical but can be found in any textbook on differential geometry. Another important observation is that $\mathcal{T}_{x(q)}(\mathcal{X})=\operatorname{span}\left(x_{u}(q), x_{v}(q)\right)=$ $\operatorname{Im}(d x)_{x(q)}$, i.e. $d x: \mathbb{R}^{2} \rightarrow \mathcal{T}_{x(q)}(\mathcal{X})$ is a linear bijective map between $\mathbb{R}^{2}$ and the tangent plane $\mathcal{T}_{x(q)}(\mathcal{X})$.

Definition (Integral on Surfaces). Let $\mathcal{X}$ be a surface, let $x: U \rightarrow V \subset \mathcal{X}$ be a chart and let $f: \mathcal{X} \rightarrow \mathbb{R}$ be a real valued function. We define the integral of $f$ on subsurface $V$ with respect to $x$ by

$$
\int_{V} f d p=\int_{U} f(x(q)) \sqrt{\operatorname{det}\left((d x)_{q}^{\top}(d x)_{q}\right)} d q
$$

The area $\mathcal{A}(V)$ of $V$ is defined by setting $f(p)=1$ everywhere, i.e. $\mathcal{A}(V)=\int_{V} 1 d p$.

Definition (Differential of Functions on Surfaces). Let $\mathcal{X}$ be a surface. The differential of some vector-valued function $f: \mathcal{X} \rightarrow \mathbb{R}^{m}$ at point $p \in \mathcal{X}$ is a function $(d f)_{p}: \mathcal{T}_{p}(\mathcal{X}) \rightarrow \mathbb{R}^{m}$ defined by

$$
\begin{equation*}
(d f)_{p}(w)=(f \circ \alpha)^{\prime}\left(s_{0}\right), \tag{1}
\end{equation*}
$$

where $\alpha:\left(s_{0}-\epsilon, s_{0}+\epsilon\right) \rightarrow \mathcal{X}, s_{0} \in \mathbb{R}$ is any regular curve, such that $\alpha\left(s_{0}\right)=p$ and $\alpha^{\prime}\left(s_{0}\right)=w$.

Loosely speaking, the differential $(d f)_{p}(w)$ is the derivative of $f$ when moving on the surface at point $p$ in direction $w$. The domain of $(d f)_{p}$ is $\mathcal{T}_{p}(\mathcal{X})$ since this is the space of all possible directions at $p$.

Exercise 2 (One Point). 1. Show that the integral over $V \subset \mathcal{X}$ is well defined, i.e. it does not depend on the choice of the chart.
2. Show that the differential of some function $f$ on a surface is well defined, i.e. it does not depend on the choice of the curve.
3. Show that $(d f)_{p}: \mathcal{T}_{p}(\mathcal{X}) \rightarrow \mathbb{R}^{m}$ is a linear map.

Hint. Recall that the differential $d f$ of some function $f: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$ between realvalued vector spaces is well defined and the chain rule $(d(f \circ g))_{a}=(d f)_{g(a)}(d g)_{a}$ can be applied. For the second part of the exercise proving $(f \circ \alpha)^{\prime}(0)=(d(f \circ$ $x))_{x^{-1}(p)}\left(d x^{-1}\right)_{p} w$ might be helpful.

Solution. 1. Let $x: U_{x} \rightarrow V, y: U_{y} \rightarrow V, V \subset \mathcal{X}$, be two charts and let $\phi: U_{y} \rightarrow U_{x}, \phi=x^{-1} \circ y$. Using integration by substitution it holds

$$
\begin{aligned}
\int_{V} f d v & =\int_{U_{x}} f(x(q)) \sqrt{\operatorname{det}\left((d x)_{q}^{\top}(d x)_{q}\right)} d q \\
& =\int_{\phi^{-1}\left(U_{x}\right)} f(x \circ \phi(q)) \sqrt{\operatorname{det}\left((d x \circ \phi)(q)^{\top}(d x \circ \phi)(q)\right)}|\operatorname{det} d \phi(q)| d q \\
& =\int_{U_{y}} f \circ x \circ \phi \sqrt{\operatorname{det}\left((d x \circ \phi)^{\top}(d x \circ \phi)\right)}|\operatorname{det} d \phi| d q
\end{aligned}
$$

The second line follows from substituting $q$ by $\phi(q)$. The third line follows from $U_{y}=\phi^{-1}\left(U_{x}\right)$ and removes the argument $q$ for better readability. Observe that the from the chainrule $d(x \circ \phi)=(d x \circ \phi) d \phi$, the invertability of $d \phi$ and $x \circ \phi=y$ it follows $d x \circ \phi=d(x \circ \phi)(d \phi)^{-1}=d y(d \phi)^{-1}$. It follows

$$
\begin{aligned}
\int_{V} f d v & =\int_{U_{y}} f \circ y \sqrt{\operatorname{det}\left(\left(d y(d \phi)^{-1}\right)^{\top} d y(d \phi)^{-1}\right)}|\operatorname{det} d \phi| d q \\
& =\int_{U_{y}} f \circ y \sqrt{\operatorname{det}\left(d y^{\top} d y\right) \operatorname{det}(d \phi)^{-2}}|\operatorname{det} d \phi| d q \\
& =\int_{U_{y}} f \circ y \sqrt{\operatorname{det}\left(d y^{\top} d y\right)} d q .
\end{aligned}
$$

2. Let $\alpha, \beta:(-\epsilon, \epsilon) \rightarrow \mathcal{X}$ be two regular curves on surface $\mathcal{X}$ with $\alpha(0)=\beta(0)=$ $p \in \mathcal{X}$ and $\alpha^{\prime}(0)=\beta^{\prime}(0)=w$. We need to show that for any $f: \mathcal{X} \rightarrow \mathbb{R}^{m}$ and any $w \in \mathcal{T}_{p}(\mathcal{X})$ it holds $(f \circ \alpha)^{\prime}(0)=(f \circ \beta)^{\prime}(0)$. (If we could apply the chain rule here, the proof would be simple. But the chain rule needs a proper definition of the differential $d f$ of $f$, which we are just about to proof.) Consider a chart $x: U \rightarrow V$, such that $\operatorname{Im}(\alpha) \cup \operatorname{Im}(\beta) \subset V$. Now we can apply the chain rule with a simple trick:

$$
\begin{aligned}
(f \circ \alpha)^{\prime}(0) & =\left(f \circ x \circ x^{-1} \circ \alpha\right)^{\prime}(0) \\
& =\left((f \circ x) \circ\left(x^{-1} \circ \alpha\right)\right)^{\prime}(0) \\
& =(d(f \circ x))_{x^{-1} \circ \alpha(0)}\left(x^{-1} \circ \alpha\right)^{\prime}(0) \\
& =(d(f \circ x))_{x^{-1}(\alpha(0))}\left(d x_{x^{-1}(\alpha(0))}\right)^{-1} \alpha^{\prime}(0) \\
& =(d(f \circ x))_{x^{-1}(p)}\left(d x_{x^{-1}(p)}\right)^{-1} w .
\end{aligned}
$$

Doing this analoguously for $\beta$ leads to $(f \circ \alpha)^{\prime}(0)=(f \circ \beta)^{\prime}(0)$.
3. We have shown that $(d f)_{p}(w)=(d(f \circ x))_{x^{-1}(p)}\left(d x_{x^{-1}(p)}\right)^{-1} w$. Since both functions $f \circ x, x^{-1}$ map from vector spaces to vector spaces, their differentials $(d(f \circ x)),(d x)^{-1}$ are linear maps between the corresponding tangent spaces. Thus ( $d f$ ) is also a linear map.

Exercise 3 (One Point). 1. Compute the following integral:

$$
\int_{0}^{1} \int_{0}^{1-u} a(1-u-v)+b u+c v d v d u
$$

2. Let $a, b, c \in \mathbb{R}^{3}$ be the corners of some triangle $T$, such that $T \subset \mathcal{X}$ is part of surface $\mathcal{X}$. Consider two functions $f, g: \mathcal{X} \rightarrow \mathbb{R}$ that are linear on $T$ and take values $f(a), g(b) \in \mathbb{R}, f(b)=f(c)=g(a)=g(c)=0$ at it's corners. Compute the integral on the interior $\operatorname{int}(T)$ of $T$ :

$$
\int_{\text {int }(T)} f(t) g(t) d t
$$

