Analysis of Three-Dimensional Shapes
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Weekly Exercises 3

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Please send in only Latex-PDF. If you have hand-written solutions, please hand them in during the lecture.

Mathematics: Curves in Space

Definition (Curves)**.** Let $I \subset \mathbb{R}$ be an open interval of $\mathbb{R}$. A *curve* is a smooth map $\alpha : I \to \mathbb{R}^3$. The curve is *regular*, if $\alpha'(s) \neq \mathbf{0}$ for any $s \in I$. The derivative $\alpha'(s)$ with respect to s is called the *tangent vector* at s or at point $\alpha(s)$.

Definition (Length of Curves)**.** Let $[a,b] \subset I$ be a compact interval and $\alpha : I \to \mathbb{R}^3$ be a regular curve. The *length* $L_a^b(\alpha)$ *of* α *from* a *to* b is defined by

$$L_a^b(\alpha) = \int_a^b \|\alpha'(s)\| \, ds = \int_a^b \sqrt{\langle \alpha'(s), \alpha'(s) \rangle} ds, \tag{1}$$

where $\|\cdot\|$ is the usual ℓ_2-norm and $\langle \cdot, \cdot \rangle$ the standard inner product of $\mathbb{R}^3$.

Definition (Rigid Motion)**.** A transformation $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is called a *rigid motion* if there exists a matrix $A \in O(n)$ and a vector $b \in \mathbb{R}^n$, such that $\phi(x) = Ax + b$ for all $x \in \mathbb{R}^n$.

Recall that $A \in O(n) \Leftrightarrow AA^\top = \mathrm{Id} \Leftrightarrow \det A = \pm 1 \Leftrightarrow \phi$ is an isometry of $\mathbb{R}^n$. Note that by this definition reflections ($\det A = -1$) are also rigid motions. Other literature might exclude reflections from the set of rigid motions.

Exercise 1 (One point)**.**

1. Let $\alpha : (-4,4) \to \mathbb{R}^3, s \mapsto (s, 2s, s^2+1)$ be a curve. Show that the curve is regular and compute the tangent vector of α at $s \in (-4,4)$ and the length $L_{-2}^2(\alpha)$.
2. Let $\alpha : (-4,4) \to \mathbb{R}^3, s \mapsto (s, 2s, s^2+1)$ be a regular curve and let $\gamma : (-2,2) \to (-4,4), s \mapsto 2s$ be a function. Show that the curve $\beta = \alpha \circ \gamma$ is regular and compute the tangent vector of β at $s \in (-2,2)$ and the length $L_{-1}^1(\beta)$.
3. Let $\alpha : I \to \mathbb{R}^3$ be a regular curve, $[a,b] \subset I$ and let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be a rigid motion. Show that $L_a^b(\phi \circ \alpha) = L_a^b(\alpha)$.

Solution. 1. The tangent vector is $\alpha'(s) = (1, 2, 2s)$ for any $s \in (-4, 4)$. It will never be zero, hence α is regular.

$$\begin{aligned}
L_{-2}^{2}(\alpha) &= \int_{-2}^{2} \sqrt{\langle \alpha'(s), \alpha'(s) \rangle} ds \\
&= \int_{-2}^{2} \sqrt{5 + 4s^2} ds \\
&= 2 \int_{-2}^{2} \sqrt{\frac{5}{4} + s^2} ds.
\end{aligned}$$

Now we look up

$$\int \sqrt{a^2 + t^2} dt = \frac{1}{2}(t\sqrt{a^2 + t^2} + a^2 \ln(t + \sqrt{a^2 + t^2}))$$

and get

$$\begin{aligned}
L_{-2}^{2}(\alpha) &= \big[t\sqrt{\frac{5}{4} + t^2} + \frac{5}{4} \ln(t + \sqrt{\frac{5}{4} + t^2})\big]_{-2}^{2} \\
&= ((2\sqrt{\frac{5}{4} + 4} + \frac{5}{4} \ln(2 + \sqrt{\frac{5}{4} + 4)}) - (-2\sqrt{\frac{5}{4} + 4} + \frac{5}{4} \ln(-2 + \sqrt{\frac{5}{4} + 4)})) \\
&= ((\sqrt{21} + \frac{5}{4} \ln(2 + \sqrt{\frac{21}{4}})) - (-\sqrt{21} + \frac{5}{4} \ln(-2 + \sqrt{\frac{21}{4}}))) \\
&= 12.5277
\end{aligned}$$

2. Since $\beta(s) = \alpha \circ \gamma(s) = (2s, 4s, 4s^2 + 1)$ the tangent vector is $\beta'(s) = (2, 4, 8s) \neq \mathbf{0}$ and hence β regular.

$$\begin{aligned}
L_{-1}^{1}(\beta) &= \int_{-1}^{1} \sqrt{\langle \beta'(s), \beta'(s) \rangle} ds \\
&= \int_{-1}^{1} \sqrt{20 + 64s^2} ds = 8 \int_{-1}^{1} \sqrt{\frac{5}{16} + s^2} ds.
\end{aligned}$$

Again using

$$\int \sqrt{a^2 + t^2} dt = \frac{1}{2}(t\sqrt{a^2 + t^2} + a^2 \ln(t + \sqrt{a^2 + t^2}))$$

we get

$$\begin{aligned}
L_{-1}^{1}(\alpha) &= 4\big[t\sqrt{\frac{5}{16} + t^2} + \frac{5}{16} \ln(t + \sqrt{\frac{5}{16} + t^2})\big]_{-1}^{1} \\
&= \big[4\sqrt{\frac{5}{16} + 1} + \frac{5}{4} \ln(1 + \sqrt{\frac{5}{16} + 1})\big] - \big[-4\sqrt{\frac{5}{16} + 1} + \frac{5}{4} \ln(-1 + \sqrt{\frac{5}{16} + 1})\big] \\
&= ((\sqrt{21} + \frac{5}{4} \ln(1 + \sqrt{\frac{21}{16}})) - (-\sqrt{21} + \frac{5}{4} \ln(-1 + \sqrt{\frac{21}{16}}))) \\
&= 12.5277.
\end{aligned}$$

3. Since ϕ is a rigid motion, we can assume there exists some $A \in \mathbb{R}^{3\times 3}, b \in \mathbb{R}^3, A^\top A = \mathrm{Id}$, such that $\phi(x) = Ax + b$. The Jacobian $\mathcal{J}_\phi$ of ϕ is $\mathcal{J}_\phi(x) = A$ at any point $x \in \mathbb{R}^3$ and derivative of $\phi \circ \alpha$ is

$$\begin{aligned}(\phi \circ \alpha)'(s) &= \mathcal{J}_\phi(\alpha(s))\alpha'(s) \\ &= A\alpha'(s).\end{aligned}$$

The proof is completed by expanding the definition of length $L_a^b(\phi \circ \alpha)$:

$$\begin{aligned}L_a^b(\phi \circ \alpha) &= \int_a^b \sqrt{\langle(\phi \circ \alpha)'(s), (\phi \circ \alpha)'(s)\rangle} \\ &= \int_a^b \sqrt{\langle A\alpha'(s), A\alpha'(s)\rangle} \\ &= \int_a^b \sqrt{\langle A^\top A\alpha'(s), \alpha'(s)\rangle} \\ &= \int_a^b \sqrt{\langle \alpha'(s), \alpha'(s)\rangle} \\ &= L_a^b(\alpha).\end{aligned}$$

Mathematics: Surfaces in Space

Definition (Surface)**.** A non-empty set $\mathcal{X} \subset \mathbb{R}^3$ is called a *surface* if, for each $p \in \mathcal{X}$, there exists an open neighbourhood $N \subset \mathbb{R}^3$, an open set $U \subset \mathbb{R}^2$ and a differentiable map $x : U \to V$, where $V = \mathcal{X} \cap N$, such that it holds:

1. $x : U \to V$ is a homeomorphism and
2. the partial derivatives $x_u(q) = \frac{\partial x}{\partial u}(q) \in \mathbb{R}^3$ and $x_v(q) = \frac{\partial x}{\partial v}(q) \in \mathbb{R}^3$ are non-zero and linearly independent for all $q \in U \subset \mathbb{R}^2$.

Recall that a homeomorphism $x : U \to V$ is a continuous and bijective map, such that the inverse x^{-1} is also continuous. The set $\mathcal{T}_p(\mathcal{X}) = \mathrm{span}(x_u(x^{-1}(p)), x_v(x^{-1}(p)))$ is a 2-dimensional subspace of $\mathbb{R}^3$ and is called the *tangent space* of $\mathcal{X}$ at point p.

The individual maps x are called *charts* or *parameterizations* and a collection of charts covering $\mathcal{X}$ is sometimes called an *atlas*.

Exercise 2 (One point)**.** Let $U = (-4, 4) \times (-4, 4) \subset \mathbb{R}^2$ and let $x : U \to \mathbb{R}^3, \binom{u}{v} \mapsto (u, v, (u + v)^2 + 1)^\top$ be a chart of surface $\mathcal{X} = \mathrm{Im}(x)$.

1. Compute the partial derivatives $x_u\binom{u}{v} = \frac{\partial x}{\partial u}\binom{u}{v}, x_v\binom{u}{v} = \frac{\partial x}{\partial v}\binom{u}{v}$ for any $\binom{u}{v} \in U$.
2. Compute the differential $(dx)_{\binom{0.5}{1}}\binom{2}{1}$ of x at point $\binom{0.5}{1} \in U$ in direction $\binom{2}{1}$.

Solution. 1. The derivatives are

$$x_u\big(\begin{smallmatrix} u \\ v \end{smallmatrix}\big) = (1, 0, 2u + 2v)^\top \text{ and } x_u\big(\begin{smallmatrix} u \\ v \end{smallmatrix}\big) = (0, 1, 2u + 2v)^\top.$$

2. The differential is

$$(dx)_{\big(\begin{smallmatrix} 0.5 \\ 1 \end{smallmatrix}\big)}\big(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\big) = (x_u\big(\begin{smallmatrix} 0.5 \\ 1 \end{smallmatrix}\big), x_v\big(\begin{smallmatrix} 0.5 \\ 1 \end{smallmatrix}\big))\big(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\big) = \big(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 3 & 3 \end{smallmatrix}\big)\big(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\big) = \big(\begin{smallmatrix} 2 \\ 1 \\ 9 \end{smallmatrix}\big).$$