Analysis of Three-Dimensional Shapes
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Weekly Exercises 7

Room: 02.09.023
Wed, 10.06.2015, 14:15-15:45
Submission deadline: Tue, 09.06.2015, 23:59 to windheus@in.tum.de
Please send in only Latex-PDF. If you have hand-written solutions, please hand them in during the lecture.

Heat Kernel

Recall the definition of the *heat kernel* from the lecture. Given a shape $\mathcal{S}$ and time parameter $t \in \mathbb{R}_{\geq 0}$, the heat kernel k_t is defined by

$$k_t(x, y) = \sum_i \exp(-\lambda_i t)\phi_i(x)\phi_i(y)$$

for any two points $x, y \in \mathcal{S}$, where $\lambda_i \in \mathbb{R}_{\geq 0}, \phi_i : \mathcal{S} \to \mathbb{R}$ are the eigenvalues and eigenfunctions of the Laplace-Beltrami operator, respectively.

Exercise 1 (One Point)**.** Show that the heat kernel satisfies the properties of a *diffusion kernel*:

1. $\forall x, y \in \mathcal{S} : k_t(x, y) = k_t(x, y)$ (symmetry),
2. $\int_{\mathcal{S}} \int_{\mathcal{S}} (k_t(x, y))^2 dxdy < \infty$ (square integrability),
3. $\int_{\mathcal{S}} \int_{\mathcal{S}} k_t(x, y) f(x) f(y) dxdy \geq 0$ for any $f : \mathcal{S} \to \mathbb{R}$ (positive semi-definiteness),
4. $\int_{\mathcal{S}} k_t(x, y) dy = 1$ for any $x \in \mathcal{S}$ (conservation).

Solution. 1. $\forall x, y \in \mathcal{S} : k_t(x, y) = k_t(x, y)$holds by definition.

2. $\int_{\mathcal{S}}\int_{\mathcal{S}}(k_t(x,y))^2dxdy < \infty$ (square integrability):

$$\begin{aligned}\int_{\mathcal{S}}\int_{\mathcal{S}}(k_t(x,y))^2dxdy &= \int_{\mathcal{S}}\int_{\mathcal{S}}(\sum_i \exp(-\lambda_i t)\phi_i(x)\phi_i(y))^2dxdy \\ &= \int_{\mathcal{S}}\int_{\mathcal{S}}\sum_{i,j}\exp(-\lambda_i t)\exp(-\lambda_j t)\phi_i(x)\phi_j(x)\phi_i(y)\phi_j(y)dxdy \\ &= \sum_{i,j}\exp(-\lambda_i t)\exp(-\lambda_j t)\int_{\mathcal{S}}\int_{\mathcal{S}}\phi_i(x)\phi_j(x)\phi_i(y)\phi_j(y)dxdy \\ &= \sum_{i,j}\exp(-\lambda_i t)\exp(-\lambda_j t)\int_{\mathcal{S}}\int_{\mathcal{S}}\phi_i(x)\phi_j(x)dx\phi_i(y)\phi_j(y)dy \\ &= \sum_{i,j}\exp(-\lambda_i t)\exp(-\lambda_j t)\int_{\mathcal{S}}\langle\phi_i,\phi_j\rangle\phi_i(y)\phi_j(y)dy \\ &= \sum_{i,j}\exp(-\lambda_i t)\exp(-\lambda_j t)\langle\phi_i,\phi_j\rangle^2 \\ &< \infty \text{ since functions } \phi_i \text{ are square integrable.}\end{aligned}$$

3. $\int_{\mathcal{S}}\int_{\mathcal{S}}k_t(x,y)f(x)f(y)dxdy \geq 0$ for any $f : \mathcal{S} \to \mathbb{R}$ (positive semi-definiteness):

$$\begin{aligned}\int_{\mathcal{S}}\int_{\mathcal{S}}k_t(x,y)f(x)f(y)dxdy &= \int_{\mathcal{S}}\int_{\mathcal{S}}(\sum_i \exp(-\lambda_i t)\phi_i(x)\phi_i(y))f(x)f(y)dxdy \\ &= \sum_i \exp(-\lambda_i t)\int_{\mathcal{S}}\int_{\mathcal{S}}\phi_i(x)\phi_i(y)f(x)f(y)dxdy \\ &= \sum_i \exp(-\lambda_i t)\langle\phi_i, f\rangle^2.\end{aligned}$$

4. $\int_{\mathcal{S}}k_t(x,y)dy = 1$ for any $x \in \mathcal{S}$ (conservation):

$$\begin{aligned}\int_{\mathcal{S}}k_t(x,y)dy &= \int_{\mathcal{S}}(\sum_i \exp(-\lambda_i t)\phi_i(x)\phi_i(y))dx \\ &= \sum_i \exp(-\lambda_i t)\phi_i(y)\int_{\mathcal{S}}\phi_i(x)dx \\ &= \exp(-\lambda_0 t)\int_{\mathcal{S}}\phi_0(y)\phi_0(x)dx \\ &= \exp(-\lambda_0 t)\int_{\mathcal{S}}\phi_0(x)^2dx \\ &= 1 \cdot 1.\end{aligned}$$