Analysis of Three-Dimensional Shapes
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Weekly Exercises 9

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Please send in only Latex-PDF. If you have hand-written solutions, please hand them in during the lecture.

Shape Differences

Let $\mathcal{M}, \mathcal{N}$ be two shapes, define $h_a^{\mathcal{M}}(f,g) = \int_{\mathcal{M}} f(p)g(p)dp, h_a^{\mathcal{N}}(f,g) = \int_{\mathcal{N}} f(p)g(p)dp$. Discretize the shapes' Laplace-Beltrami operators using the usual finite element method resulting in mass matrices $M_{\mathcal{M}}, M_{\mathcal{N}}$ and stiffness matrices $W_{\mathcal{M}}, W_{\mathcal{N}}$, such that $\Phi_{\mathcal{M}}, \Phi_{\mathcal{N}}$ are the eigen functions (written as columns of the matrices) and $\Lambda_{\mathcal{M}}, \Lambda_{\mathcal{N}}$ are their eigen values (written as diagonal matrices). Recall that $\Phi_{\mathcal{M}}$ is orthonormal with respect to $M_{\mathcal{M}}$, i.e. $\Phi_{\mathcal{M}}^\top M_{\mathcal{M}} \Phi_{\mathcal{M}} = \text{Id}$.

Exercise 1 (One Point)**.**

1. Let F be a linear transformation (functional map) between $L^2(\mathcal{M})$ and $L^2(\mathcal{N})$, represented by matrix C mapping functions in the eigen basis $\Phi_{\mathcal{M}}$ onto functions in the eigen basis $\Phi_{\mathcal{N}}$. The Riesz representation theorem tells us we can write $h_a^{\mathcal{N}}(F(f), F(g)) = h_a^{\mathcal{M}}(f, D(g))$, for any functions $f, g \in L^2(\mathcal{M})$, where $D : L^2(\mathcal{M}) \to L^2(\mathcal{M})$ is a linear map. Show that D is self adjoint, i.e. $h_a^{\mathcal{M}}(f, D(g)) = h_a^{\mathcal{M}}(D(f), g)$ for all f, g.
2. Now assume F is area preserving and D is represented by matrix V in the eigen basis Φ_M, show that $V = C^\top C$.
3. Define $h_c^{\mathcal{M}}(f,g) = \int_{\mathcal{M}} \langle \nabla f(p), \nabla g(p) \rangle dp, h_c^{\mathcal{N}}(f,g) = \int_{\mathcal{N}} \langle \nabla f(p), \nabla g(p) \rangle dp$. Again let F be a linear transformation (functional map) between $L^2(\mathcal{M})$ and $L^2(\mathcal{N})$, represented by matrix C mapping functions in the eigen basis $\Phi_{\mathcal{M}}$ onto functions in the eigen basis $\Phi_{\mathcal{N}}$. Let $f, g \in L^2(\mathcal{M})$, such that $f = \sum_i \phi_i \alpha_i, g = \sum_i \phi_i \beta_i$. Show that $h_c^{\mathcal{N}}(F(f), F(g))$ can be represented by $h_c^{\mathcal{N}}(F(f), F(g)) = \alpha^\top R \beta$, where R is some matrix such that $\Lambda_{\mathcal{M}} R = C^\top \Lambda_{\mathcal{N}} C$.
4. Assume F is conformal, what can you say about R?

Solution.

1. Let us start by using the symmetry of the definition of $h_a^{\mathcal{N}}(\cdot, \cdot)$:

$$h_a^{\mathcal{N}}(F(f), F(g)) = h_a^{\mathcal{N}}(F(g), F(f)) \tag{1}$$
$$h_a^{\mathcal{M}}(f, D(g)) = h_a^{\mathcal{M}}(g, D(f)) \tag{2}$$
$$h_a^{\mathcal{M}}(f, D(g)) = h_a^{\mathcal{M}}(D(f), g). \tag{3}$$

The second line follows from the Riesz representation theorem and the third from the the symmetry of the definition of $h_a^{\mathcal{M}}(\cdot, \cdot)$, again.

2. We know that $h_a^{\mathcal{N}}(f,g) = \langle \alpha, \beta \rangle$ if α, β are the corresponding eigen coefficients of f, g. From the representation theorem we get

$$\langle \alpha, V\beta \rangle = h_a^{\mathcal{M}}(f, D(g)) = h_a^{\mathcal{N}}(F(f), F(g)) = \langle C\alpha, C\beta \rangle = \langle \alpha, C^\top C\beta \rangle, \quad (4)$$

i.e. $V = C^\top C$. Area preservation means that

$$h_a^{\mathcal{N}}(F(f), F(g)) = h_a^{\mathcal{M}}(f, g). \quad (5)$$

Written in eigen coefficients:

$$\langle \alpha, C^\top C\beta \rangle = \langle \alpha, \mathrm{Id}\,\beta \rangle, \quad (6)$$

for any α, β. Leading to $C^\top C = \mathrm{Id}$.

3. We can rewrite h_c with the help of the Laplacian into

$$h_c(f,g) = \int \langle \nabla f(p), \nabla g(p) \rangle dp = \int \langle f(p), -\Delta g(p) \rangle dp. \quad (7)$$

If f, g are written in their eigen coefficients α, β we have

$$h_c(f,g) = \langle \alpha, \Lambda\beta \rangle. \quad (8)$$

Combining this with the representation theorem we get

$$h_c^{\mathcal{M}}(f, D(g)) = h_c^{\mathcal{N}}(F(g), F(f)) \quad (9)$$

$$\langle \alpha, \Lambda_{\mathcal{M}} R\beta \rangle = \langle C\alpha, \Lambda_{\mathcal{N}} C\beta \rangle = \langle \alpha, C^\top \Lambda_{\mathcal{N}} C\beta \rangle. \quad (10)$$

Since the last line has to hold for any α, β, the matrix R is a matrix for which equation

$$\Lambda_{\mathcal{M}} R = C^\top \Lambda_{\mathcal{N}} C \quad (11)$$

holds.

4. If F is conformal, it holds $h_a^{\mathcal{M}}(f,g) = h_a^{\mathcal{N}}(F(f), F(g))$ or written in eigen coefficients $\langle \alpha, \Lambda_{\mathcal{M}}\beta \rangle = \langle \alpha, C^\top \Lambda_{\mathcal{N}} C\beta \rangle$. Since this should be satisfied for any f, g, the equation $\Lambda_{\mathcal{M}} = C^\top \Lambda_{\mathcal{N}} C$ must hold.

Programming: Shape Differences

Exercise 2 (Two points)**.** Download and expand the file `ex9.zip` from the lecture website. Modifiy the file `findAnaloguousShape.m` to implement the function as explained below. You need one data set of shapes to test implementation. Choose and download one data set from `http://people.csail.mit.edu/drdaniel/mesh_animation/index.html` (one of the `meshes.tgz` files linked on the page). The meshes in those data sets are registered. Corresponding vertices on two meshes have the same index on both meshes. You can run the script `exercise.m` to test and visualize your solutions after you specified the correct path to the shape data set.

`function [meshIndex] = findAnaloguousShape(M1, M2, M3, meshFilenames, k)` accepts three meshes `M1`, `M2`, `M3` and a cell array of filenames `meshFilenames`. `k` is the number of eigen functions. It returns the index of the mesh that relates best to `M3` as `M2` relates to `M1`. Use the formula shown on page 22 from the lecture on *shape differences* to quantify shape analogy.