Chapter 1
Convex Analysis

Nonlinear Multiscale Methods for Image and Signal Analysis
SS 2015

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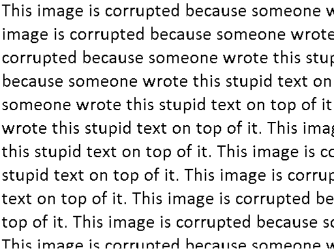
Variational Problems

Example: Inpainting



$$\hat{u} = \arg\min_{u \in \mathbb{R}^n} \left\| \sqrt{(D_x u)^2 + (D_y u)^2} \right\|_1, \quad \text{such that } u_i = f_i \ \forall i \in I$$

with index set I of uncorrupted pixels.

Variational Problems

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Convexity

Variational Problems

Let us repeat some basics things to talk about

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Definition

- For $E : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$, we call

  $$\text{dom}(E) := \{u \in \mathbb{R}^n \mid E(u) < \infty\}$$

  the domain of E.
- We call E proper if $\text{dom}(E) \neq \emptyset$.

Variational Problems

Definition: Convex Function

We call $E : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ a convex function if

1. $\text{dom}(E)$ is a convex set, i.e. for all $u, v \in \text{dom}(E)$ and all $\theta \in [0,1]$ it holds that $\theta u + (1-\theta)v \in \text{dom}(E)$.

2. For all $u, v \in \text{dom}(E)$ and all $\theta \in [0,1]$ it holds that

$$E(\theta u + (1-\theta)v) \leq \theta E(u) + (1-\theta)E(v)$$

We call E strictly convex, if the inequality in 2 is strict for all $\theta \in]0,1[$, and $v \neq u$.

Variational Problems

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with index set I of uncorrupted pixels.

$\rightarrow$ Discuss convexity.

Existence

Variational Problems

When does

$$\hat{u} = \arg\min_{u \in \mathbb{R}^n} E(u)$$

exist?

Variational Problems

When does

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exist?

- E is lower semi-continuous, i.e. for all u

$$\liminf_{v \to u} E(v) \geq E(u)$$

  holds.
- There exists an α such that

$$\{u \mid E(u) \leq \alpha\}$$

  is non-empty and bounded.

Proof: Board.

Variational Problems

Fundamental Theorem of Optimization

If $E : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is lower semi-continuous and has a nonempty bounded sublevelset, then there exists

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Variational Problems

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Remark: For a proper convex function, lower semi-continuity is the same as the closedness of the sublevelsets.

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Remark: For a proper convex function, lower semi-continuity is the same as the closedness of the sublevelsets.

Examples on the board:

- A convex continuous function that does not have a minimizer
- A convex function with bounded sublevelsets that does not have a minimizer

Variational Problems

Continuity of Convex Functions

If $E : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is convex, then E is locally Lipschitz (and hence continuous) on int(dom(E)).

Proof: Exercise (in 1d)

Board: Considering the interior is important!

Variational Problems

Continuity of Convex Functions

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Board: Considering the interior is important!

Conclusion

If $E : \mathbb{R}^n \to \mathbb{R}$ is convex, then E is continuous.

Variational Problems

Definition

We call $E : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ coercive, if all sequences $(u_n)_n$ with $\|u_n\| \to \infty$ meet $E(u_n) \to \infty$.

Theorem

If $E : \mathbb{R}^n \to \mathbb{R}$ is convex and coercive, then there exists

$$\hat{u} = \arg\min_{u \in \mathbb{R}^n} E(u).$$

Variational Problems

When is

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Variational Problems

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Theorem

If $E : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is convex, then any local minimum is a global minimum. If E is strictly convex, the global minimum is unique.

Subdifferential Calculus

Variational Problems

What is an optimality condition for

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Definition: Subdifferential

We call

$$\partial E(u) = \{p \in \mathbb{R}^n \mid E(v) - E(u) - \langle p, v - u \rangle \geq 0\}$$

the subdifferential of E at u.

- Elements of $\partial E(u)$ are called subgradients.
- If $\partial E(u) \neq \emptyset$, we call E subdifferentiable at E.
- By convention, $\partial E(u) = \emptyset$ for $u \neq \text{dom}(E)$.

Variational Problems

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Theorem: Optimality condition

Let $0 \in \partial E(\hat{u})$. Then $\hat{u} \in \arg\min_u E(u)$.

Variational Problems

Examples:

- The ℓ^1 norm.

Variational Problems

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- The ℓ^1 norm.
- Functional

$$E(u) = \begin{cases} 0 & \text{if } u \geq 0 \\ \infty & \text{else.} \end{cases}$$

Variational Problems

Definition: Relative Interior

The *relative interior* of a convex set M is defined as

$$\mathrm{ri}(M) := \{x \in M \mid \forall y \in M,\ \exists \lambda > 1,\ \text{s.t. } \lambda x + (1-\lambda) y \in M\}$$

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Theorem

If E is a proper convex function and $u \in \text{ri}(\text{dom}(E))$, then $\partial E(u)$ is not empty.

Variational Problems

Theorem: Sum rule

Let E_1, E_2 be convex functions such that

$$\operatorname{ri}(\operatorname{dom}(E_1)) \cap \operatorname{ri}(\operatorname{dom}(E_2)) \neq \emptyset,$$

then it holds that

$$\partial(E_1 + E_2)(u) = \partial E_1(u) + \partial E_2(u).$$

Variational Problems

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Example: Minimize $(u - f)^2 + \iota_{u \geq 0}(u)$.

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Example: Minimize $(u-f)^2 + \iota_{u \geq 0}(u)$.

Example: Minimize $0.5(u-f)^2 + \alpha|u|$.

Variational Problems

Theorem: Chain rule

If $A \in \mathbb{R}^{m\times n}$, $E : \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ is convex, and $\text{ri}(\text{dom}(E)) \cap \text{range}(A) \neq \emptyset$, then

$$\partial(E \circ A)(u) = A^* \partial E(Au)$$

Variational Problems

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Example: Minimize $\|Au - f\|_2^2$.