Chapter 2
Multiscale Methods

Nonlinear Multiscale Methods for Image and Signal Analysis
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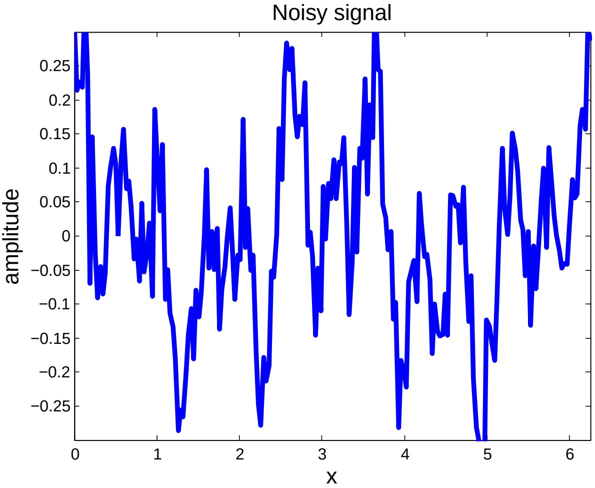
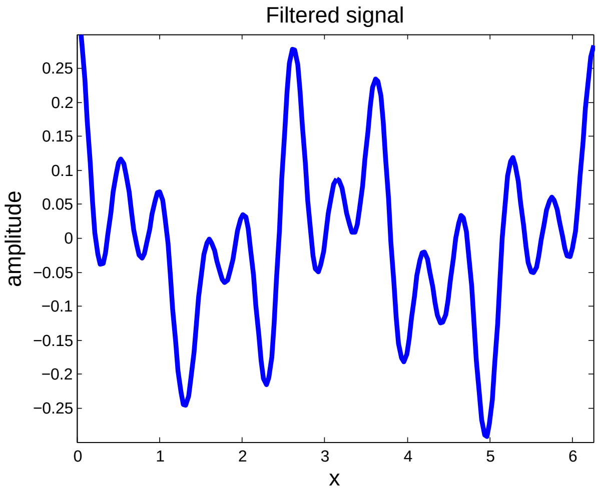
Linear image and signal filtering

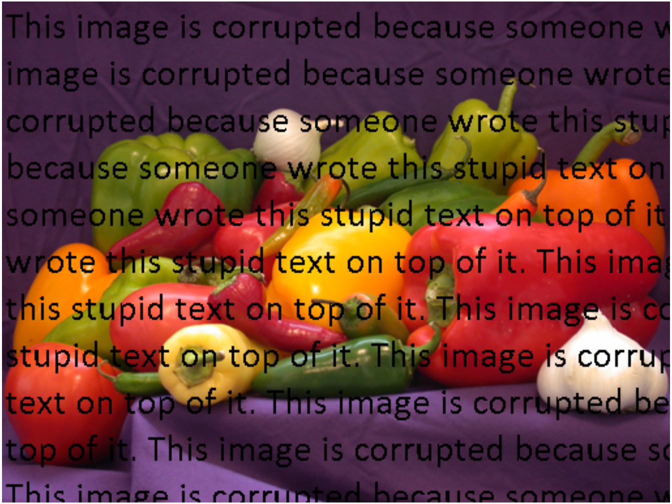
Linear signal denoising



Linear signal denoising



Linear image inpainting



Linear image inpainting

Linear image deblurring

Linear image deblurring

Linear image denoising

Linear image denoising

Linear image sharpening

Linear image sharpening

Linear image and signal filtering

How can we understand the behavior of linear filters?

Consider for instance the simple linear sharpening

$$\hat{u} = imfilter(f, k) = k * f$$

with a kernel

$$k = fspecial('unsharp') = \begin{bmatrix} -0.1667 & -0.6667 & -0.1667 \\ -0.6667 & 4.3333 & -0.6667 \\ -0.1667 & -0.6667 & -0.1667 \end{bmatrix}.$$

Remember the Convolution Theorem:

$$\hat{u} = k * f \;\Rightarrow\; \mathcal{F}(\hat{u}) = \mathcal{F}(k)\mathcal{F}(f)$$

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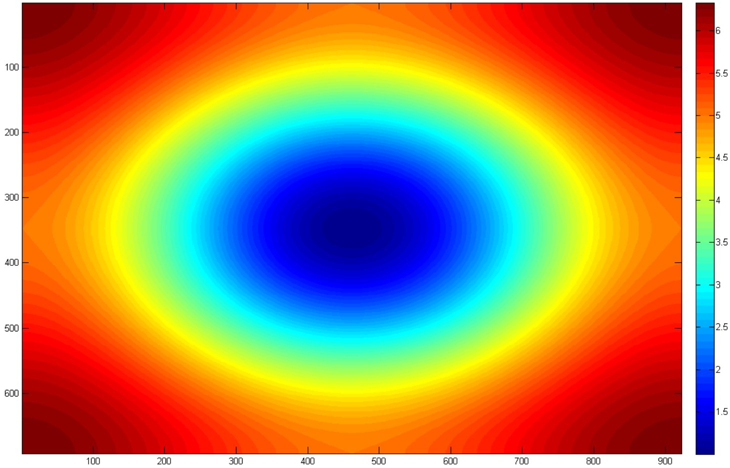
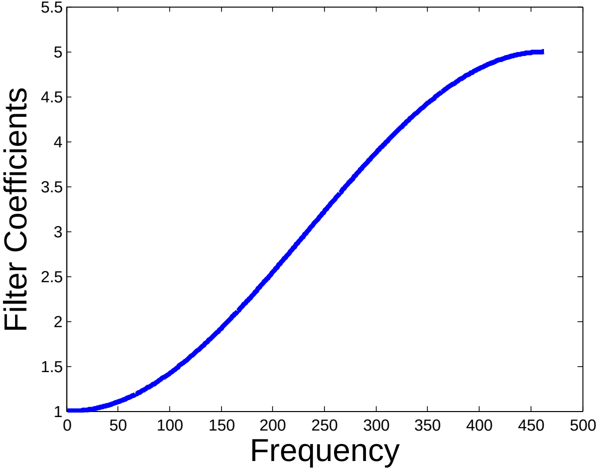
Absolute values of $\mathcal{F}(k)$.



Middle is 1 and corresponds to the lowest frequency

Linear image and signal filtering

Or in 1d:



Linear image and signal filtering

This representation is very intuitive for us, since we have an understanding of frequencies and can look at filters.

But what does it mean mathematically?

What does $\mathcal{F}(\hat{u}) = \mathcal{F}(k)\mathcal{F}(f)$ do?

Pointwise (or componentwise) multiplication
$\to \mathcal{F}(k)$ is diagonal!

Linear image and signal filtering

Let us go back to linear algebra:

Consider

$$\hat{u} = Af$$

for some symmetric positive definite matrix $A \in \mathbb{R}^{n \times n}$.

Note that any linear operator can be written in this form!

There exists an orthonormal basis $\{v_1, ..., v_n\}$ of eigenvectors of A with eigenvalues $\{\lambda_1, ..., \lambda_n\}$:

$$Av_i = \lambda_i v_i$$

Linear image and signal filtering

We write

$$f = \sum_i a_i v_i.$$

Now

$$\hat{u} = Af = \sum_i a_i A v_i = \sum_i \lambda_i a_i v_i.$$

Let us represent $\hat{u}$ in the eigenbasis of A and denote its coefficients by b_i. Then

$$\begin{pmatrix} b_1 \\ . \\ . \\ b_n \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & . & 0 \\ . & . & . & . \\ . & . & . & . \\ 0 & . & 0 & \lambda_n \end{pmatrix} \begin{pmatrix} a_1 \\ . \\ . \\ a_n \end{pmatrix}$$

$\rightarrow$ We have diagonalized A and you know this since > 3 years.

Linear image and signal filtering

$$\begin{pmatrix} b_1 \\ . \\ . \\ b_n \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & . & 0 \\ . & . & . & . \\ . & . & . & . \\ 0 & . & 0 & \lambda_n \end{pmatrix} \begin{pmatrix} a_1 \\ . \\ . \\ a_n \end{pmatrix}$$

Engineering interpretation:

- λ_i is the filter coefficients for the *i*-th *frequency*.
- $\lambda_i > 1$ means boosting the *frequency*, $\lambda_i < 1$ means damping the frequency.
- The interpretation of the *frequency* is given by the eigenvector v_i.
- Any convolution diagonalizes under sin/cos, which yields a classical frequency.
- Other linear operators lead to other meanings of frequencies.

Linear image and signal filtering

Variational methods can be linear, too...

$$\hat{u} = \arg\min_u \frac{1}{2}\|u - f\|_2^2 + \alpha\|\nabla u\|_2^2. \tag{1}$$

Optimality at

$$0 = \hat{u} - f - \alpha\Delta\hat{u},$$

or

$$\hat{u} = (I - \alpha\Delta)^{-1} f.$$

- Depends linearly on f.
- Also diagonalizes via FFT.
- Variational method (1) is nothing but a special frequency filter...
- ... and does not work very well.

From linear to nonlinear image filtering

Now consider

$$\hat{u} = \arg\min_{u} \frac{1}{2}\|u - f\|_2^2 + \alpha\|\nabla u\|_1, \tag{2}$$

which is highly nonlinear.

Absolutely no eigenvector theory!

Or is there?