Chapter 2
Multiscale Methods

Nonlinear Multiscale Methods for Image and Signal Analysis
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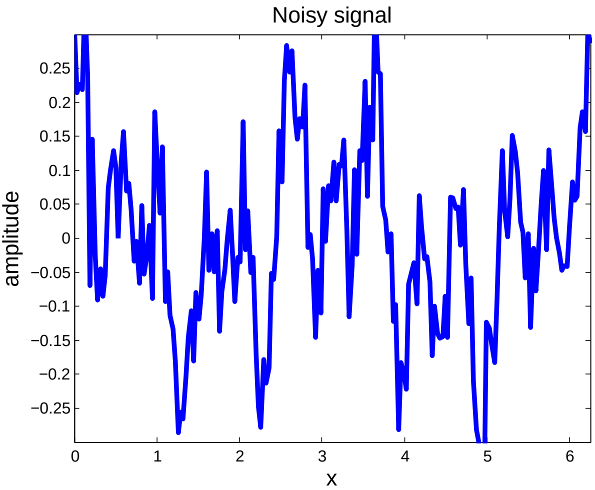
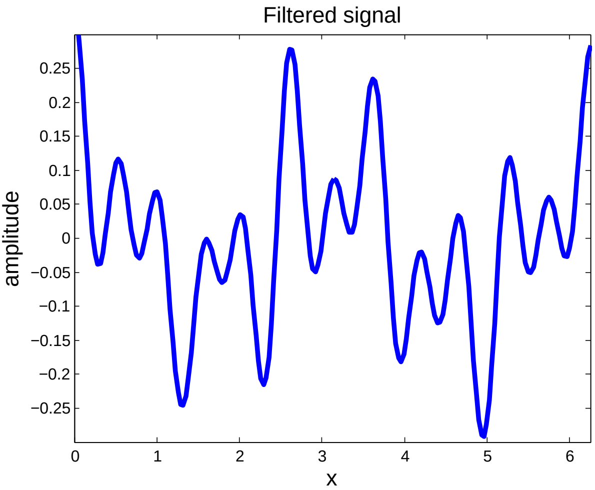
Linear image and signal filtering

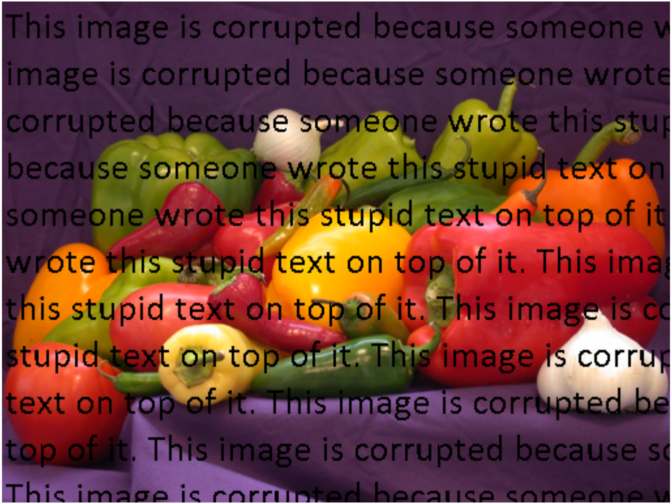
Linear signal denoising



Linear signal denoising



Linear image inpainting



Linear image inpainting

Linear image deblurring

Linear image deblurring

Linear image denoising

Linear image denoising

Linear image sharpening

Linear image sharpening

Linear image and signal filtering

How can we understand the behavior of linear filters?

Consider for instance the simple linear sharpening

$$\hat{u} = imfilter(f, k) = k * f$$

with a kernel

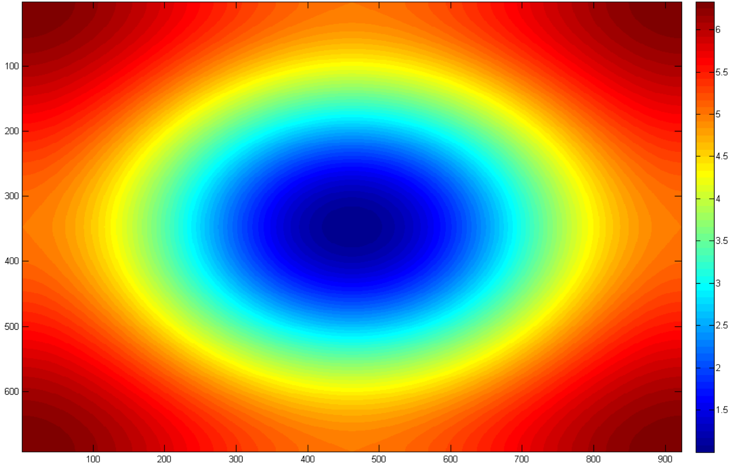
$$k = fspecial('unsharp') = \begin{bmatrix} -0.1667 & -0.6667 & -0.1667 \\ -0.6667 & 4.3333 & -0.6667 \\ -0.1667 & -0.6667 & -0.1667 \end{bmatrix}.$$

Remember the Convolution Theorem:

$$\hat{u} = k * f \;\Rightarrow\; \mathcal{F}(\hat{u}) = \mathcal{F}(k)\mathcal{F}(f)$$

Linear image and signal filtering

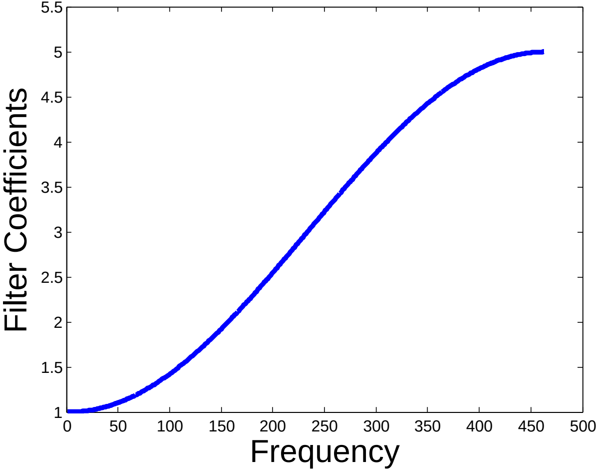
Absolute values of $\mathcal{F}(k)$.



Middle is 1 and corresponds to the lowest frequency

Linear image and signal filtering

Or in 1d:



Linear image and signal filtering

This representation is very intuitive for us, since we have an understanding of frequencies and can look at filters.

But what does it mean mathematically?

What does $\mathcal{F}(\hat{u}) = \mathcal{F}(k)\mathcal{F}(f)$ do?

Pointwise (or componentwise) multiplication
$\rightarrow \mathcal{F}(k)$ is diagonal!

Linear image and signal filtering

Let us go back to linear algebra:

Consider

$$\hat{u} = Af$$

for some symmetric positive definite matrix $A \in \mathbb{R}^{n \times n}$.

Note that any linear operator can be written in this form!

There exists an orthonormal basis $\{v_1, ..., v_n\}$ of eigenvectors of A with eigenvalues $\{\lambda_1, ..., \lambda_n\}$:

$$Av_i = \lambda_i v_i$$

Linear image and signal filtering

We write

$$f = \sum_i a_i v_i.$$

Now

$$\hat{u} = Af = \sum_i a_i A v_i = \sum_i \lambda_i a_i v_i.$$

Let us represent $\hat{u}$ in the eigenbasis of A and denote its coefficients by b_i. Then

$$\begin{pmatrix} b_1 \\ . \\ . \\ b_n \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & . & 0 \\ . & . & . & . \\ . & . & . & . \\ 0 & . & 0 & \lambda_n \end{pmatrix} \begin{pmatrix} a_1 \\ . \\ . \\ a_n \end{pmatrix}$$

$\rightarrow$ We have diagonalized A and you know this since > 3 years.

Linear image and signal filtering

$$\begin{pmatrix} b_1 \\ . \\ . \\ b_n \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & . & 0 \\ . & . & . & . \\ . & . & . & . \\ 0 & . & 0 & \lambda_n \end{pmatrix} \begin{pmatrix} a_1 \\ . \\ . \\ a_n \end{pmatrix}$$

Engineering interpretation:

- λ_i is the filter coefficients for the *i*-th *frequency*.
- $\lambda_i > 1$ means boosting the *frequency*, $\lambda_i < 1$ means damping the frequency.
- The interpretation of the *frequency* is given by the eigenvector v_i.
- Any convolution diagonalizes under sin/cos, which yields a classical frequency.
- Other linear operators lead to other meanings of frequencies.

Linear image and signal filtering

Variational methods can be linear, too...

$$\hat{u} = \arg\min_u \frac{1}{2}\|u - f\|_2^2 + \alpha\|\nabla u\|_2^2. \tag{1}$$

Optimality at

$$0 = \hat{u} - f - \alpha\Delta\hat{u},$$

or

$$\hat{u} = (I - \alpha\Delta)^{-1} f.$$

- Depends linearly on f.
- Also diagonalizes via FFT.
- Variational method (1) is nothing but a special frequency filter...
- ... and does not work very well.

From linear to nonlinear image filtering

Now consider

$$\hat{u} = \arg\min_u \frac{1}{2}\|u - f\|_2^2 + \alpha\|\nabla u\|_1, \tag{2}$$

which is highly nonlinear.

Absolutely no eigenvector theory!

Or is there?

Nonlinear Spectral Theory[1]

[1] Largely based on: M. Benning and M. Burger, *Ground States and Singular Vectors of Convex Variational Regularization Methods*, 2013

Linear Variational Model

Let us start with the (general) previous observation that

$$\hat{u} = \arg\min_u \frac{1}{2}\|u - f\|_2^2 + \frac{\alpha}{2}\|Ku\|_2^2,$$

leads to

$$(I + \alpha K'K)\hat{u} = f$$

such that the singular vectors v of the above problem are the eigenvectors of the symmetric, positive semi-definite matrix $K'K$, i.e. there exist $v_\lambda \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ such that

$$\lambda v_\lambda = K'Kv_\lambda$$

or

$$\lambda v_\lambda \in \partial J(v_\lambda)$$

for $J(u) = \frac{1}{2}\|Ku\|^2$.

Nonlinear Variational Model

The variational model

$$\hat{u} = \arg\min_u \frac{1}{2}\|u - f\|_2^2 + \frac{\alpha}{2}\|Ku\|_2^2,$$

leads to the singular vector description that there exist $v_\lambda \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ such that

$$\lambda v_\lambda \in \partial J(v_\lambda).$$

The latter makes sense for any convex regularization!

Can we study general J, e.g. TV regularization?

One-homogeneous functionals

Notation

From now on we denote the set of all proper, convex, lower semi-continuous functions $J : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ by $\Gamma_0(\mathbb{R}^n)$.

Definition: One-homogeneous

We call $J \in \Gamma_0(\mathbb{R}^n)$ (absolutely) 1-homogeneous, if

$$J(\lambda u) = |\lambda| J(u)$$

holds for all $\lambda \in \mathbb{R}$.

Example: $J(u) = \|Ku\|$ is one-homogeneous for any norm.

One-homogeneous functionals

Triangle inequality of 1-homogeneous functions

Let $J \in \Gamma_0(\mathbb{R}^n)$ be 1-hom. Then J meets the triangle inequality.

Domain of 1-homogeneous functions

Let $J \in \Gamma_0(\mathbb{R}^n)$ be 1-homogeneous. Then $\mathrm{dom}(J)$ is a linear subspace.

Convention: Domain of 1-homogeneous functions

Without restriction of generality (for variational problems), we will assume that any 1-homogeneous $J \in \Gamma_0(\mathbb{R}^n)$ maps from $\mathbb{R}^n$ to $\mathbb{R}$.

Remark: Using the above convention we conclude that such a J is continuous and defines a semi-norm on $\mathbb{R}^n$.

One-homogeneous functionals

Kernel of 1-homogeneous functions

Let $J \in \Gamma_0(\mathbb{R}^n)$ be 1-homogeneous. Then

$$\ker(J) = \{u \in \mathbb{R}^n \mid J(u) = 0\}$$

is a linear subspace.

Remark: J defines a norm on $\ker(J)^\perp$.

Nonnegativity of 1-homogeneous functions

Let $J \in \Gamma_0(\mathbb{R}^n)$ be 1-hom. Then $J(0) = 0$ and $J(u) \geq 0$ for all u.

One-homogeneous functionals

Subdifferential of 1-homogeneous functions

Let $J \in \Gamma_0(\mathbb{R}^n)$ be 1-hom. and subdifferentiable at u. Then

$$\partial J(u) = \{p \in \mathbb{R}^n \mid J(u) = \langle p, u \rangle,\ J(v) \geq \langle p, v \rangle \ \forall v \in \mathbb{R}^n\}$$

0-homogeneous subdifferential

Let $J \in \Gamma_0(\mathbb{R}^n)$ be 1-hom. and subdifferentiable at u. Then

$$\partial J(au) = \partial J(u)$$

holds for all $a > 0$.

Generalized singular vectors

Let us return to

$$\lambda u \in \partial J(u).$$

What about normalization?

- Linear case: If v_λ meets $\lambda v_\lambda \in \partial J(v_\lambda)$, then $v = a v_\lambda$ meets $\lambda v \in \partial J(v)$, too.
- One-homogeneous: If v_λ meets $\lambda v_\lambda \in \partial J(v_\lambda)$, then $v = a v_\lambda$ meets $\frac{\lambda}{a} v \in \partial J(v)$.

Generalized singular vectors

Definition: Generalized singular vector

For $J \in \Gamma_0(\mathbb{R}^n)$ we call a v_λ with $\|v_\lambda\|_2 = 1$ a singular vector of J with singular value $\lambda \in \mathbb{R}$ if

$$\lambda v_\lambda \in \partial J(v_\lambda).$$

Observations for J being one-homogeneous:

- If there exists a v_λ with

  $$\lambda v_\lambda \in \partial J(v_\lambda)$$

  then $\tilde{v}_\lambda = \frac{v_\lambda}{\|v_\lambda\|_2}$ is a singular vector to J.
- For a singular value λ it holds that $\lambda = J(v_\lambda) \geq 0$.
- Smaller singular values correspond to smaller "frequencies".

Generalized singular vectors

What about the existence of singular vectors?

Generalized singular vectors

Definition: Ground States

Let $J \in \Gamma_0(\mathbb{R}^n)$ be 1-homogeneous. A *ground state* of J is defined by

$$u_0 = \arg \min_{\substack{u \in \ker(J)^\perp \\ \|u\|_2 = 1}} J(u).$$

Ground states exist

Let $J \in \Gamma_0(\mathbb{R}^n)$ be 1-homogeneous, and let $\ker(J) + \mathrm{dom}(J)^\perp \neq \mathbb{R}^n$. Then a ground state exists.

Proof: Board.

Generalized singular vectors

Ground states are singular vectors

Let $J \in \Gamma_0(\mathbb{R}^n)$ be one-homogeneous with ground state u_0. Then u_0 is a singular vector with the singular value $\lambda_0 = J(u_0)$.

Proof: Board

Remark: A ground state is a singular vector with the smallest possible singular values: $\lambda \geq \lambda_0$ for all singular values λ.

Generalized singular vectors

Example: 1d TV