



# Chapter 2

## Multiscale Methods

*Nonlinear Multiscale Methods for Image and Signal Analysis*  
SS 2015

Linear Filtering

Spectral Theory

Nonlinear singular vectors

1-homogeneous functions

Existence questions

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# Linear image and signal filtering

# Linear signal denoising

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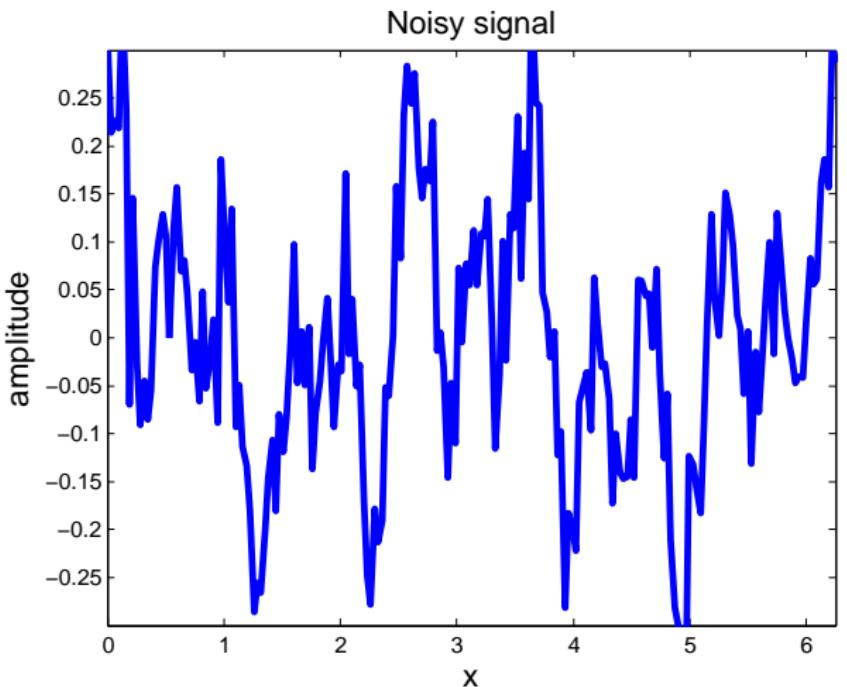
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# Linear signal denoising

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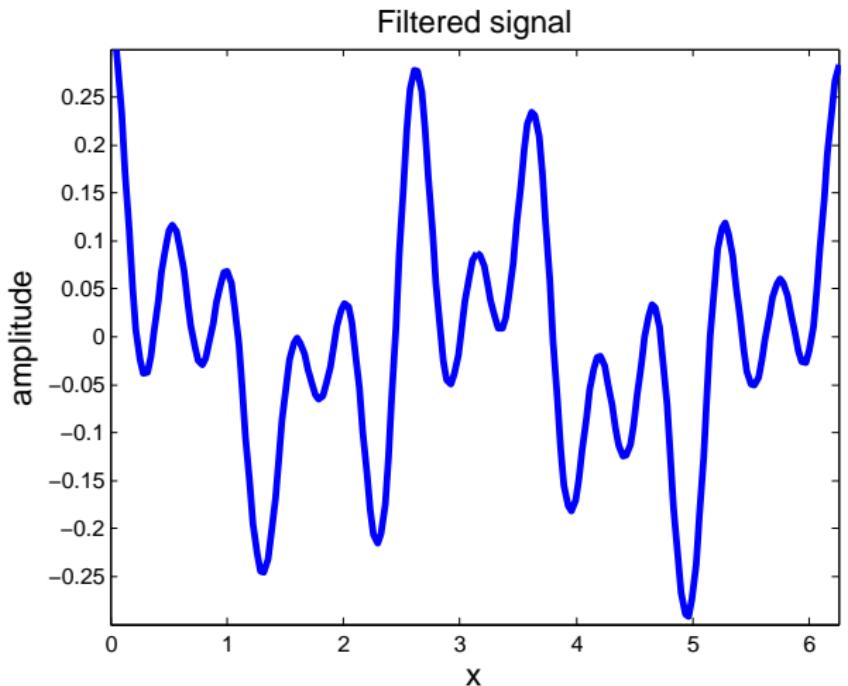
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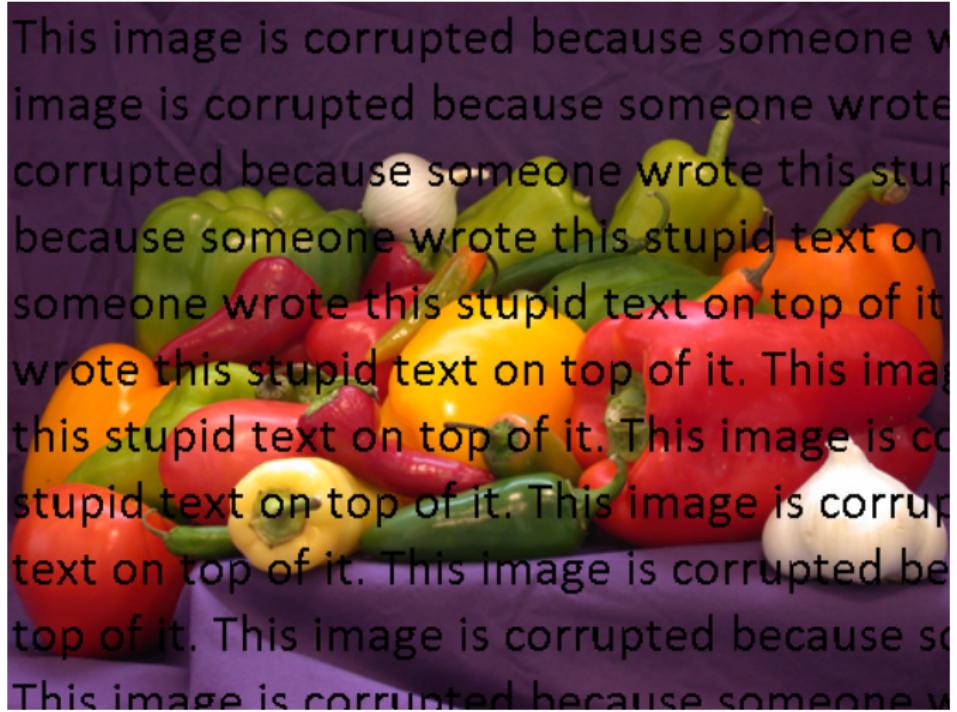
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# Linear image inpainting

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# Linear image sharpening

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# Linear image sharpening

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## How can we understand the behavior of linear filters?

Consider for instance the simple linear sharpening

$$\hat{u} = \text{imfilter}(f, k) = k * f$$

with a kernel

$$k = \text{fspecial('unsharp')} = \begin{bmatrix} -0.1667 & -0.6667 & -0.1667 \\ -0.6667 & 4.3333 & -0.6667 \\ -0.1667 & -0.6667 & -0.1667 \end{bmatrix}.$$

Remember the Convolution Theorem:

$$\hat{u} = k * f \Rightarrow \mathcal{F}(\hat{u}) = \mathcal{F}(k)\mathcal{F}(f)$$

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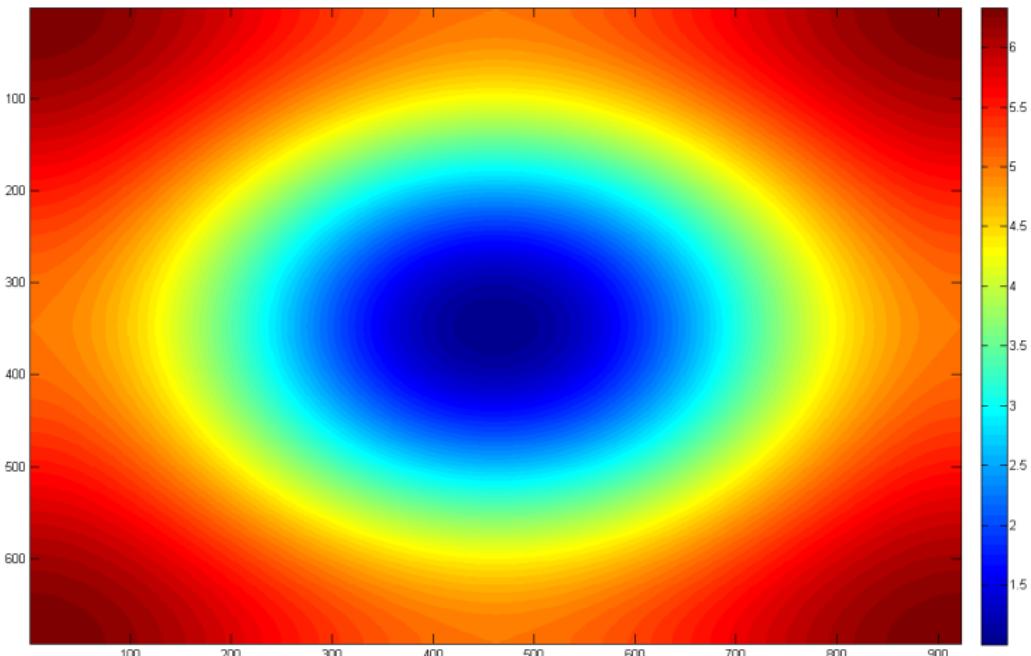
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Absolute values of  $\mathcal{F}(k)$ .



Middle is 1 and corresponds to the lowest frequency



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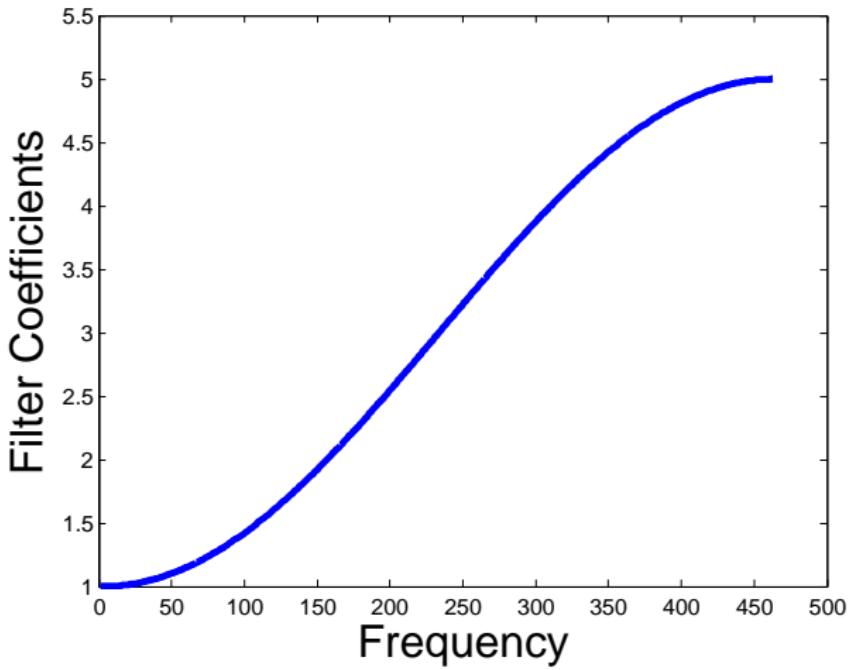
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Or in 1d:





This representation is very intuitive for us, since we have an understanding of frequencies and can look at filters.

But what does it mean mathematically?

What does  $\mathcal{F}(\hat{u}) = \mathcal{F}(k)\mathcal{F}(f)$  do?

Pointwise (or componentwise) multiplication  
→  $\mathcal{F}(k)$  is diagonal!

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Let us go back to linear algebra:

Consider

$$\hat{u} = Af$$

for some symmetric positive definite matrix  $A \in \mathbb{R}^{n \times n}$ .

Note that any linear operator can be written in this form!

There exists an orthonormal basis  $\{v_1, \dots, v_n\}$  of eigenvectors of  $A$  with eigenvalues  $\{\lambda_1, \dots, \lambda_n\}$ :

$$Av_i = \lambda_i v_i$$



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# Linear image and signal filtering

We write

$$f = \sum_i a_i v_i.$$

Now

$$\hat{u} = Af = \sum_i a_i A v_i = \sum_i \lambda_i a_i v_i.$$

Let us represent  $\hat{u}$  in the eigenbasis of  $A$  and denote its coefficients by  $b_i$ . Then

$$\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ddots & 0 & \lambda_n \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

→ We have diagonalized  $A$  and you know this since > 3 years.

$$\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_n \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

Engineering interpretation:

- $\lambda_i$  is the filter coefficients for the *i-th frequency*.
- $\lambda_i > 1$  means boosting the *frequency*,  $\lambda_i < 1$  means damping the frequency.
- The interpretation of the *frequency* is given by the eigenvector  $v_i$ .
- Any convolution diagonalizes under sin/cos, which yields a classical frequency.
- Other linear operators lead to other meanings of frequencies.



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## Linear image and signal filtering

Variational methods can be linear, too...

$$\hat{u} = \arg \min_u \frac{1}{2} \|u - f\|_2^2 + \alpha \|\nabla u\|_2^2. \quad (1)$$

Optimality at

$$0 = \hat{u} - f - \alpha \Delta \hat{u},$$

or

$$\hat{u} = (I - \alpha \Delta)^{-1} f.$$

- Depends linearly on  $f$ .
- Also diagonalizes via FFT.
- Variational method (1) is nothing but a special frequency filter...
- ... and does not work very well.

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Now consider

$$\hat{u} = \arg \min_u \frac{1}{2} \|u - f\|_2^2 + \alpha \|\nabla u\|_1, \quad (2)$$

which is highly nonlinear.

Absolutely no eigenvector theory!

**Or is there?**

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# Nonlinear Spectral Theory<sup>1</sup>

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<sup>1</sup>Largely based on: M. Benning and M. Burger, *Ground States and Singular Vectors of Convex Variational Regularization Methods*, 2013

Let us start with the (general) previous observation that

$$\hat{u} = \arg \min_u \frac{1}{2} \|u - f\|_2^2 + \frac{\alpha}{2} \|Ku\|_2^2,$$

leads to

$$(I + \alpha K' K) \hat{u} = f$$

such that the singular vectors  $v$  of the above problem are the eigenvectors of the symmetric, positive semi-definite matrix  $K' K$ , i.e. there exist  $v_\lambda \in \mathbb{R}^n$ ,  $\lambda \in \mathbb{R}$  such that

$$\lambda v_\lambda = K' K v_\lambda$$

or

$$\lambda v_\lambda \in \partial J(v_\lambda)$$

for  $J(u) = \frac{1}{2} \|Ku\|^2$ .



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The variational model

$$\hat{u} = \arg \min_u \frac{1}{2} \|u - f\|_2^2 + \frac{\alpha}{2} \|Ku\|_2^2,$$

leads to the singular vector description that there exist  $v_\lambda \in \mathbb{R}^n$ ,  $\lambda \in \mathbb{R}$  such that

$$\lambda v_\lambda \in \partial J(v_\lambda).$$

The latter makes sense for any convex regularization!

**Can we study general  $J$ , e.g. TV regularization?**

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## Notation

From now on we denote the set of all proper, convex, lower semi-continuous functions  $J : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$  by  $\Gamma_0(\mathbb{R}^n)$ .

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## Definition: One-homogeneous

We call  $J \in \Gamma_0(\mathbb{R}^n)$  (absolutely) 1-homogeneous, if

$$J(\lambda u) = |\lambda| J(u)$$

holds for all  $\lambda \in \mathbb{R}$ .

Example:  $J(u) = \|Ku\|$  is one-homogeneous for any norm.



## One-homogeneous functionals

### Triangle inequality of 1-homogeneous functions

Let  $J \in \Gamma_0(\mathbb{R}^n)$  be 1-hom. Then  $J$  meets the triangle inequality.

### Domain of 1-homogeneous functions

Let  $J \in \Gamma_0(\mathbb{R}^n)$  be 1-homogeneous. Then  $\text{dom}(J)$  is a linear subspace.

### Convention: Domain of 1-homogeneous functions

Without restriction of generality (for variational problems), we will assume that any 1-homogeneous  $J \in \Gamma_0(\mathbb{R}^n)$  maps from  $\mathbb{R}^n$  to  $\mathbb{R}$ .

Remark: Using the above convention we conclude that such a  $J$  is continuous and defines a semi-norm on  $\mathbb{R}^n$ .

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## Kernel of 1-homogeneous functions

Let  $J \in \Gamma_0(\mathbb{R}^n)$  be 1-homogeneous. Then

$$\ker(J) = \{u \in \mathbb{R}^n \mid J(u) = 0\}$$

is a linear subspace.

Remark:  $J$  defines a norm on  $\ker(J)^\perp$ .

## Nonnegativity of 1-homogeneous functions

Let  $J \in \Gamma_0(\mathbb{R}^n)$  be 1-hom. Then  $J(0) = 0$  and  $J(u) \geq 0$  for all  $u$ .

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## Subdifferential of 1-homogeneous functions

Let  $J \in \Gamma_0(\mathbb{R}^n)$  be 1-hom. and subdifferentiable at  $u$ . Then

$$\partial J(u) = \{p \in \mathbb{R}^n \mid J(u) = \langle p, u \rangle, J(v) \geq \langle p, v \rangle \forall v \in \mathbb{R}^n\}$$

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## 0-homogeneous subdifferential

Let  $J \in \Gamma_0(\mathbb{R}^n)$  be 1-hom. and subdifferentiable at  $u$ . Then

$$\partial J(au) = \partial J(u)$$

holds for all  $a > 0$ .



Let us return to

$$\lambda u \in \partial J(u).$$

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What about normalization?

- Linear case: If  $v_\lambda$  meets  $\lambda v_\lambda \in \partial J(v_\lambda)$ , then  $v = av_\lambda$  meets  $\lambda v \in \partial J(v)$ , too.
- One-homogeneous: If  $v_\lambda$  meets  $\lambda v_\lambda \in \partial J(v_\lambda)$ , then  $v = av_\lambda$  meets  $\frac{\lambda}{a}v \in \partial J(v)$ .



# Generalized singular vectors

## Definition: Generalized singular vector

For  $J \in \Gamma_0(\mathbb{R}^n)$  we call a  $v_\lambda$  with  $\|v_\lambda\|_2 = 1$  a singular vector of  $J$  with singular value  $\lambda \in \mathbb{R}$  if

$$\lambda v_\lambda \in \partial J(v_\lambda).$$

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Observations for  $J$  being one-homogeneous:

- If there exists a  $v_\lambda$  with

$$\lambda v_\lambda \in \partial J(v_\lambda)$$

then  $\tilde{v}_\lambda = \frac{v_\lambda}{\|v_\lambda\|_2}$  is a singular vector to  $J$ .

- For a singular value  $\lambda$  it holds that  $\lambda = J(v_\lambda) \geq 0$ .
- Smaller singular values correspond to smaller “frequencies”.



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What about the existence of singular vectors?



## Definition: Ground States

Let  $J \in \Gamma_0(\mathbb{R}^n)$  be 1-homogeneous. A *ground state* of  $J$  is defined by

$$u_0 = \arg \min_{\substack{u \in \ker(J)^\perp \\ \|u\|_2=1}} J(u).$$

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## Ground states exist

Let  $J \in \Gamma_0(\mathbb{R}^n)$  be 1-homogeneous, and let  $\ker(J) + \text{dom}(J)^\perp \neq \mathbb{R}^n$ . Then a ground state exists.

Proof: Board.



## Ground states are singular vectors

Let  $J \in \Gamma_0(\mathbb{R}^n)$  be one-homogeneous with ground state  $u_0$ .  
Then  $u_0$  is a singular vector with the singular value  $\lambda_0 = J(u_0)$ .

Proof: Board

Remark: A ground state is a singular vector with the smallest possible singular values:  $\lambda \geq \lambda_0$  for all singular values  $\lambda$ .

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## Generalized singular vectors

Examples:

- Ground states for  $J(u) = \|u\|_1$ .
- Singular vectors for  $J(u) = \|u\|_1$ .

Conclusion: Ground states do not need to be unique.

- Singular vectors for 1d-TV:  $J(u) = \|D_x u\|_1$ .

$$u_1 = 1/c$$

$$u_i = \begin{cases} u_{i-1} + 1/c & \text{for } i \leq c \\ u_{i-1} - 1/c & \text{for } c < i \leq n \end{cases}$$

Then  $(D_x)^T u \in \mathbb{R}^n$  is a singular vector.

# Generalized singular vectors

Example: 1d TV



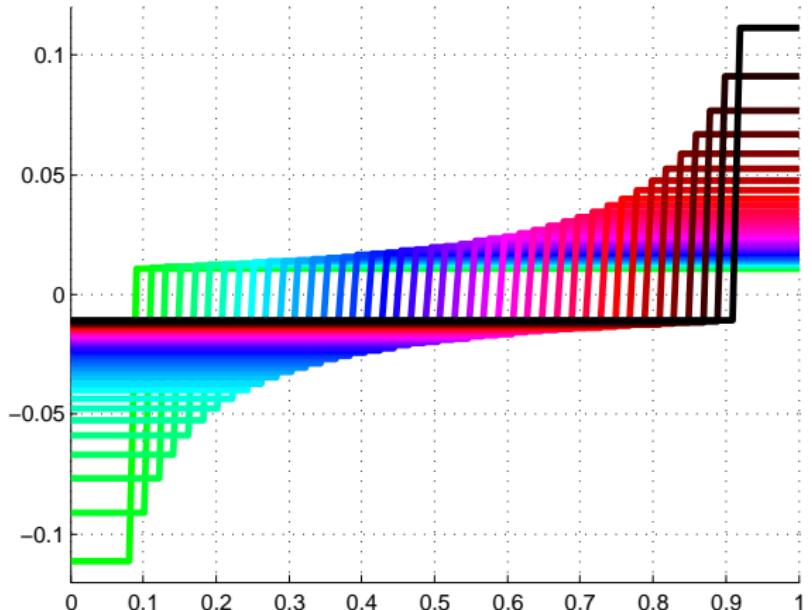
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# Generalized singular vectors

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Example: 2d TV



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Usual behavior in the linear case: For symmetric positive semi-definite matrices, there exists an orthonormal basis of singular vectors.

$$\lambda_1 \langle v_1, v_2 \rangle = \langle Av_1, v_2 \rangle = \langle v_1, Av_2 \rangle = \lambda_2 \langle v_1, v_2 \rangle$$

How about the generalized case?

→ Singular vectors to different singular values are NOT necessarily orthonormal! E.g.  $\ell^1$ .

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Can we always find an orthonormal basis consisting of singular vectors?

Linear case,  $J(u) = \frac{1}{2} \|Ku\|_2^2$ : Rayleigh Principle

Given  $\{u_1, \dots, u_n\}$ , compute

$$u_{n+1} = \arg \min_{\substack{u \in \ker(J)^\perp \\ \|u\|_2=1 \\ u \in \text{span}(u_1, \dots, u_n)^\perp}} J(u).$$

One can show: The  $u_n$  are singular vectors of  $K$  and form an orthonormal system. After completing with an orthonormal system that spans  $\ker(K)$ , one obtains an orthonormal basis.

Does the same strategy work in the nonlinear case?

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## Failure of the Rayleigh Principle

Consider

$$J(u) = \|Ku\|_1 \quad \text{with} \quad K = \begin{pmatrix} 1 & -0.1 \\ 0 & 1 \end{pmatrix}$$

Easy to verify:  $u_0 = (1, 0)^T$  is a ground state.

Orthogonal:  $u_1 = (0, 1)^T$ , but for  $v = (1, 10)^T$  we have

$$\langle J(u_1)u_1, v \rangle = 1.1 \cdot 10 = 11$$

but  $J(v) = 10$ .

$\Rightarrow u_1 = (0, 1)^T$  cannot be a singular vector.

This does not mean there is no basis of singular vectors!