



# Chapter 2

## Multiscale Methods

*Nonlinear Multiscale Methods for Image and Signal Analysis*

SS 2015

Linear Filtering

Spectral Theory

Nonlinear singular vectors

1-homogeneous functions

Existence questions

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Variational methods

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# Linear image and signal filtering

# Linear signal denoising

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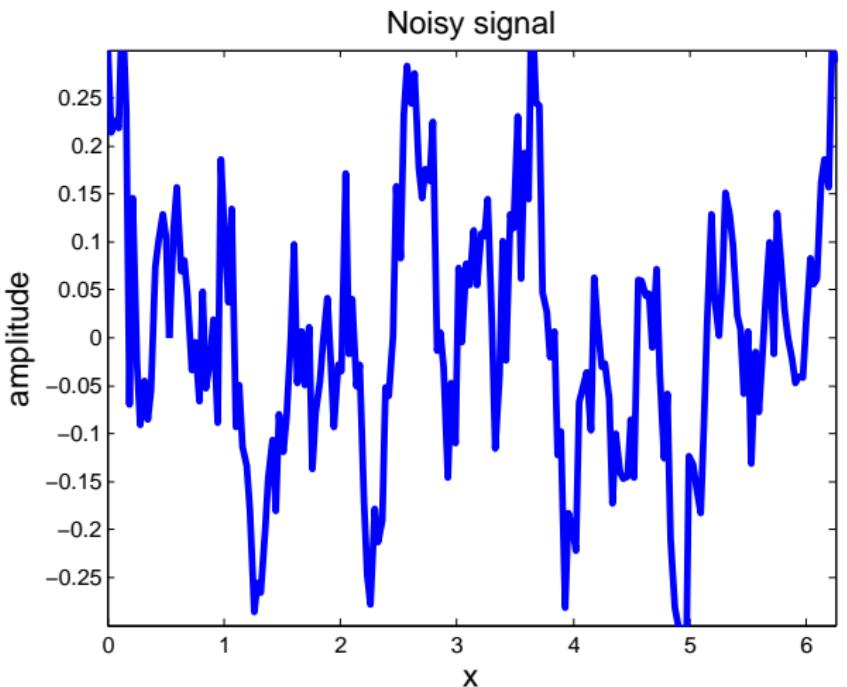
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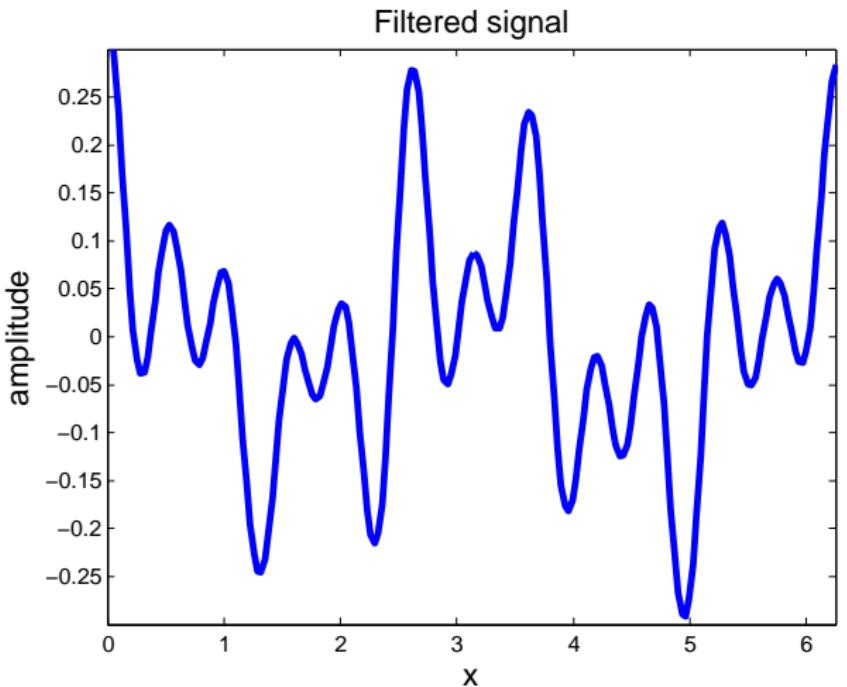
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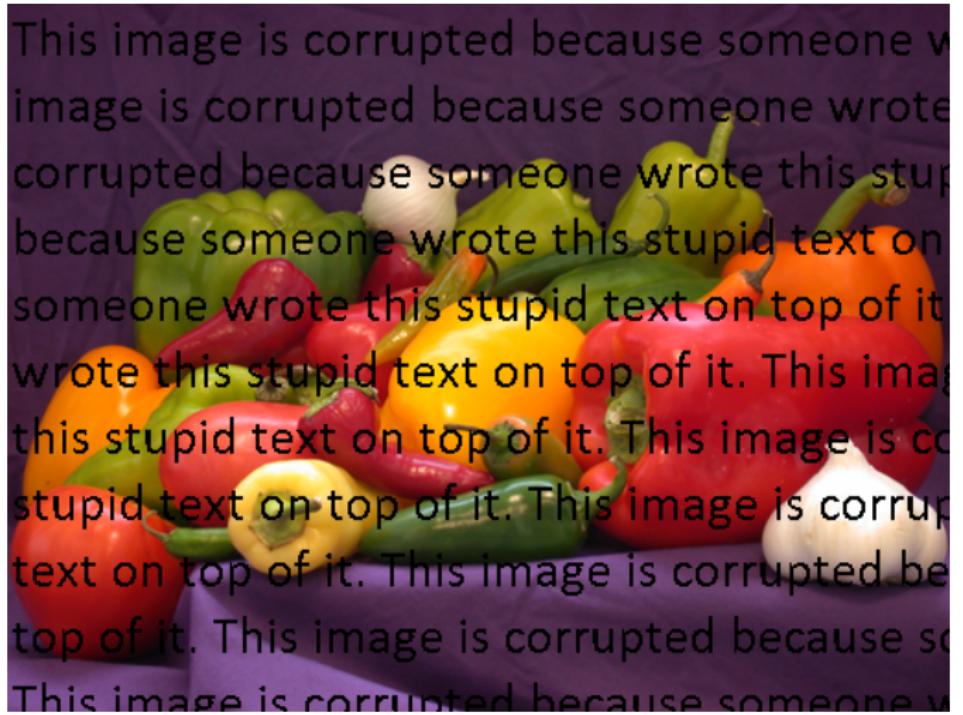
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## How can we understand the behavior of linear filters?

Consider for instance the simple linear sharpening

$$\hat{u} = \text{imfilter}(f, k) = k * f$$

with a kernel

$$k = \text{fspecial('unsharp')} = \begin{bmatrix} -0.1667 & -0.6667 & -0.1667 \\ -0.6667 & 4.3333 & -0.6667 \\ -0.1667 & -0.6667 & -0.1667 \end{bmatrix}.$$

Remember the Convolution Theorem:

$$\hat{u} = k * f \Rightarrow \mathcal{F}(\hat{u}) = \mathcal{F}(k)\mathcal{F}(f)$$

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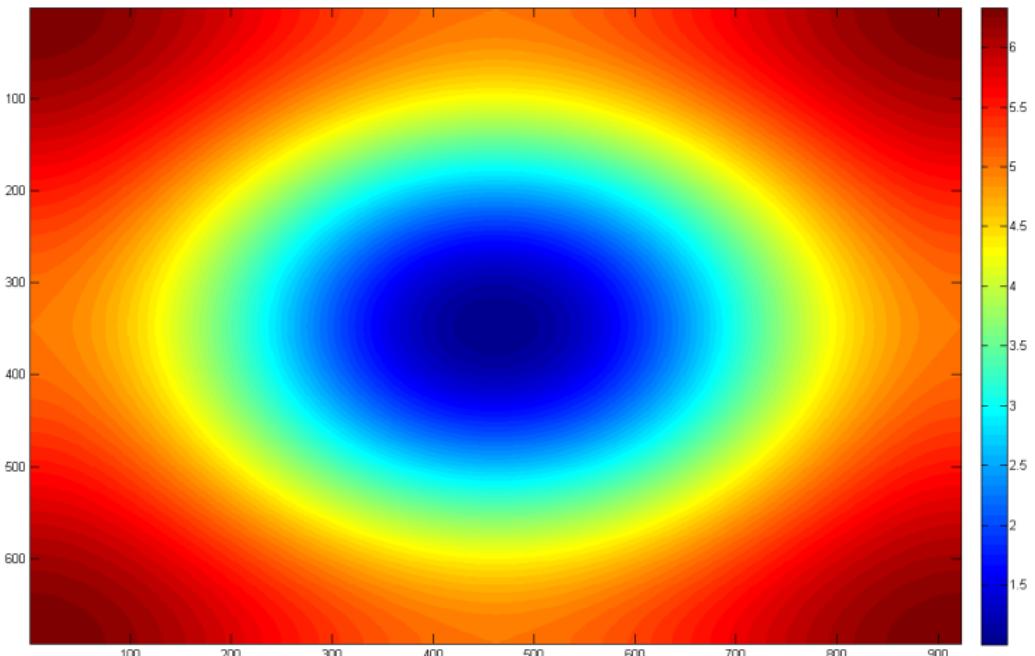
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Absolute values of  $\mathcal{F}(k)$ .



Middle is 1 and corresponds to the lowest frequency



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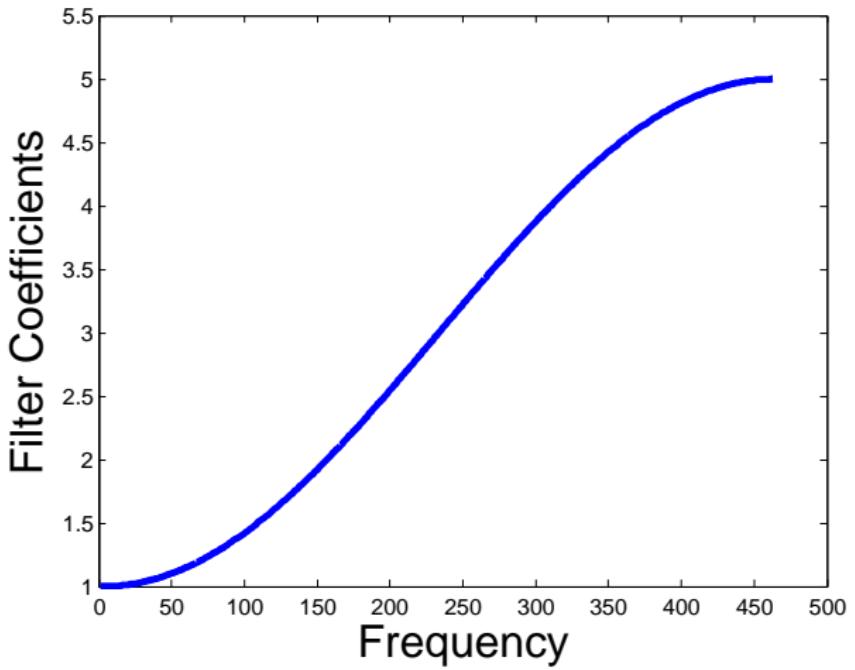
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Or in 1d:



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This representation is very intuitive for us, since we have an understanding of frequencies and can look at filters.

But what does it mean mathematically?

What does  $\mathcal{F}(\hat{u}) = \mathcal{F}(k)\mathcal{F}(f)$  do?

Pointwise (or componentwise) multiplication  
→  $\mathcal{F}(k)$  is diagonal!

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Let us go back to linear algebra:

Consider

$$\hat{u} = Af$$

for some symmetric positive definite matrix  $A \in \mathbb{R}^{n \times n}$ .

Note that any linear operator can be written in this form!

There exists an orthonormal basis  $\{v_1, \dots, v_n\}$  of eigenvectors of  $A$  with eigenvalues  $\{\lambda_1, \dots, \lambda_n\}$ :

$$Av_i = \lambda_i v_i$$



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We write

$$f = \sum_i a_i v_i.$$

Now

$$\hat{u} = Af = \sum_i a_i A v_i = \sum_i \lambda_i a_i v_i.$$

Let us represent  $\hat{u}$  in the eigenbasis of  $A$  and denote its coefficients by  $b_i$ . Then

$$\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ \cdot & \cdot & \ddots & \cdot \\ \cdot & \cdot & \ddots & \cdot \\ 0 & \cdot & 0 & \lambda_n \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

→ We have diagonalized  $A$  and you know this since > 3 years.



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$$\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_n \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

Engineering interpretation:

- $\lambda_i$  is the filter coefficients for the *i-th frequency*.
- $\lambda_i > 1$  means boosting the *frequency*,  $\lambda_i < 1$  means damping the frequency.
- The interpretation of the *frequency* is given by the eigenvector  $v_i$ .
- Any convolution diagonalizes under sin/cos, which yields a classical frequency.
- Other linear operators lead to other meanings of frequencies.



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Variational methods can be linear, too...

$$\hat{u} = \arg \min_u \frac{1}{2} \|u - f\|_2^2 + \alpha \|\nabla u\|_2^2. \quad (1)$$

Optimality at

$$0 = \hat{u} - f - \alpha \Delta \hat{u},$$

or

$$\hat{u} = (I - \alpha \Delta)^{-1} f.$$

- Depends linearly on  $f$ .
- Also diagonalizes via FFT.
- Variational method (1) is nothing but a special frequency filter...
- ... and does not work very well.



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Now consider

$$\hat{u} = \arg \min_u \frac{1}{2} \|u - f\|_2^2 + \alpha \|\nabla u\|_1, \quad (2)$$

which is highly nonlinear.

Absolutely no eigenvector theory!

**Or is there?**

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# Nonlinear Spectral Theory<sup>1</sup>

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<sup>1</sup>Largely based on: M. Benning and M. Burger, *Ground States and Singular Vectors of Convex Variational Regularization Methods*, 2013



Let us start with the (general) previous observation that

$$\hat{u} = \arg \min_u \frac{1}{2} \|u - f\|_2^2 + \frac{\alpha}{2} \|Ku\|_2^2,$$

leads to

$$(I + \alpha K' K) \hat{u} = f$$

such that the singular vectors  $v$  of the above problem are the eigenvectors of the symmetric, positive semi-definite matrix  $K' K$ , i.e. there exist  $v_\lambda \in \mathbb{R}^n$ ,  $\lambda \in \mathbb{R}$  such that

$$\lambda v_\lambda = K' K v_\lambda$$

or

$$\lambda v_\lambda \in \partial J(v_\lambda)$$

for  $J(u) = \frac{1}{2} \|Ku\|^2$ .

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The variational model

$$\hat{u} = \arg \min_u \frac{1}{2} \|u - f\|_2^2 + \frac{\alpha}{2} \|Ku\|_2^2,$$

leads to the singular vector description that there exist  $v_\lambda \in \mathbb{R}^n$ ,  $\lambda \in \mathbb{R}$  such that

$$\lambda v_\lambda \in \partial J(v_\lambda).$$

The latter makes sense for any convex regularization!

**Can we study general  $J$ , e.g. TV regularization?**

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## Notation

From now on we denote the set of all proper, convex, lower semi-continuous functions  $J : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$  by  $\Gamma_0(\mathbb{R}^n)$ .

## Definition: One-homogeneous

We call  $J \in \Gamma_0(\mathbb{R}^n)$  (absolutely) 1-homogeneous, if

$$J(\lambda u) = |\lambda| J(u)$$

holds for all  $\lambda \in \mathbb{R}$ .

Example:  $J(u) = \|Ku\|$  is one-homogeneous for any norm.

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## Triangle inequality of 1-homogeneous functions

Let  $J \in \Gamma_0(\mathbb{R}^n)$  be 1-hom. Then  $J$  meets the triangle inequality.

## Domain of 1-homogeneous functions

Let  $J \in \Gamma_0(\mathbb{R}^n)$  be 1-homogeneous. Then  $\text{dom}(J)$  is a linear subspace.

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## Convention: Domain of 1-homogeneous functions

Without restriction of generality (for variational problems), we will assume that any 1-homogeneous  $J \in \Gamma_0(\mathbb{R}^n)$  maps from  $\mathbb{R}^n$  to  $\mathbb{R}$ .

Remark: Using the above convention we conclude that such a  $J$  is continuous and defines a semi-norm on  $\mathbb{R}^n$ .



## Kernel of 1-homogeneous functions

Let  $J \in \Gamma_0(\mathbb{R}^n)$  be 1-homogeneous. Then

$$\ker(J) = \{u \in \mathbb{R}^n \mid J(u) = 0\}$$

is a linear subspace.

Remark:  $J$  defines a norm on  $\ker(J)^\perp$ .

## Nonnegativity of 1-homogeneous functions

Let  $J \in \Gamma_0(\mathbb{R}^n)$  be 1-hom. Then  $J(0) = 0$  and  $J(u) \geq 0$  for all  $u$ .

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## Subdifferential of 1-homogeneous functions

Let  $J \in \Gamma_0(\mathbb{R}^n)$  be 1-hom. and subdifferentiable at  $u$ . Then

$$\partial J(u) = \{p \in \mathbb{R}^n \mid J(u) = \langle p, u \rangle, J(v) \geq \langle p, v \rangle \forall v \in \mathbb{R}^n\}$$

## 0-homogeneous subdifferential

Let  $J \in \Gamma_0(\mathbb{R}^n)$  be 1-hom. and subdifferentiable at  $u$ . Then

$$\partial J(au) = \partial J(u)$$

holds for all  $a > 0$ .

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Let us return to

$$\lambda u \in \partial J(u).$$

What about normalization?

- Linear case: If  $v_\lambda$  meets  $\lambda v_\lambda \in \partial J(v_\lambda)$ , then  $v = av_\lambda$  meets  $\lambda v \in \partial J(v)$ , too.
- One-homogeneous: If  $v_\lambda$  meets  $\lambda v_\lambda \in \partial J(v_\lambda)$ , then  $v = av_\lambda$  meets  $\frac{\lambda}{a}v \in \partial J(v)$ .

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## Definition: Generalized singular vector

For  $J \in \Gamma_0(\mathbb{R}^n)$  we call a  $v_\lambda$  with  $\|v_\lambda\|_2 = 1$  a singular vector of  $J$  with singular value  $\lambda \in \mathbb{R}$  if

$$\lambda v_\lambda \in \partial J(v_\lambda).$$

Observations for  $J$  being one-homogeneous:

- If there exists a  $v_\lambda$  with

$$\lambda v_\lambda \in \partial J(v_\lambda)$$

then  $\tilde{v}_\lambda = \frac{v_\lambda}{\|v_\lambda\|_2}$  is a singular vector to  $J$ .

- For a singular value  $\lambda$  it holds that  $\lambda = J(v_\lambda) \geq 0$ .
- Smaller singular values correspond to smaller “frequencies”.



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## What about the existence of singular vectors?



## Definition: Ground States

Let  $J \in \Gamma_0(\mathbb{R}^n)$  be 1-homogeneous. A *ground state* of  $J$  is defined by

$$u_0 = \arg \min_{\substack{u \in \ker(J)^\perp \\ \|u\|_2=1}} J(u).$$

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## Ground states exist

Let  $J \in \Gamma_0(\mathbb{R}^n)$  be 1-homogeneous, and let  $\ker(J) + \text{dom}(J)^\perp \neq \mathbb{R}^n$ . Then a ground state exists.

Proof: Board.



## Ground states are singular vectors

Let  $J \in \Gamma_0(\mathbb{R}^n)$  be one-homogeneous with ground state  $u_0$ .  
Then  $u_0$  is a singular vector with the singular value  $\lambda_0 = J(u_0)$ .

Proof: Board

Remark: A ground state is a singular vector with the smallest possible singular values:  $\lambda \geq \lambda_0$  for all singular values  $\lambda$ .

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Examples:

- Ground states for  $J(u) = \|u\|_1$ .
- Singular vectors for  $J(u) = \|u\|_1$ .

Conclusion: Ground states do not need to be unique.

- Singular vectors for 1d-TV:  $J(u) = \|D_x u\|_1$ .

$$u_1 = 1/c$$

$$u_i = \begin{cases} u_{i-1} + 1/c & \text{for } i \leq c \\ u_{i-1} - 1/c & \text{for } c < i \leq n \end{cases}$$

Then  $(D_x)^T u \in \mathbb{R}^n$  is a singular vector.

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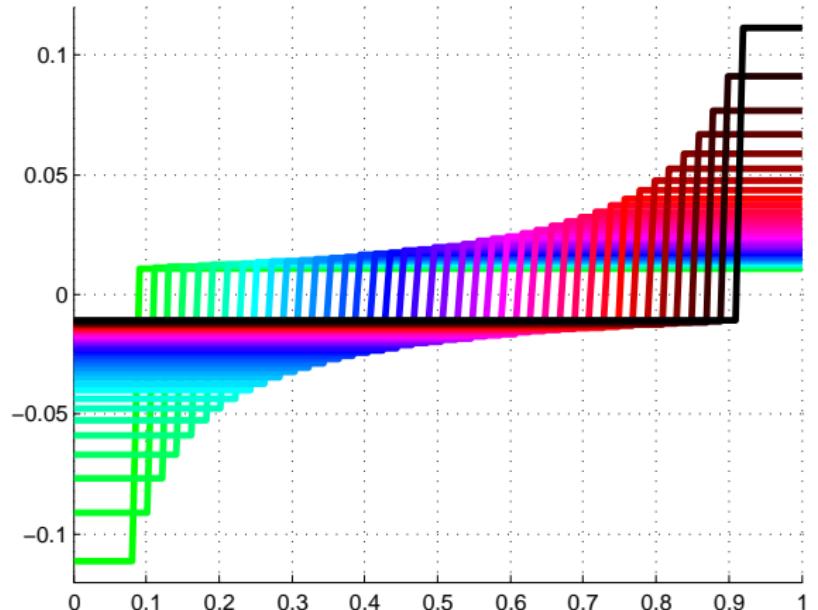
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Example: 1d TV



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Example: 2d TV



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Usual behavior in the linear case: For symmetric positive semi-definite matrices, there exists an orthonormal basis of singular vectors.

$$\lambda_1 \langle v_1, v_2 \rangle = \langle Av_1, v_2 \rangle = \langle v_1, Av_2 \rangle = \lambda_2 \langle v_1, v_2 \rangle$$

How about the generalized case?

→ Singular vectors to different singular values are NOT necessarily orthonormal! E.g.  $\ell^1$ .

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Can we always find an orthonormal basis consisting of singular vectors?

Linear case,  $J(u) = \frac{1}{2} \|Ku\|_2^2$ : Rayleigh Principle

Given  $\{u_1, \dots, u_n\}$ , compute

$$u_{n+1} = \arg \min_{\substack{u \in \ker(J)^\perp \\ \|u\|_2=1 \\ u \in \text{span}(u_1, \dots, u_n)^\perp}} J(u).$$

One can show: The  $u_n$  are singular vectors of  $K$  and form an orthonormal system. After completing with an orthonormal system that spans  $\ker(K)$ , one obtains an orthonormal basis.

Does the same strategy work in the nonlinear case?

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# Failure of the Rayleigh Principle

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Consider

$$J(u) = \|Ku\|_1 \quad \text{with} \quad K = \begin{pmatrix} 1 & -0.1 \\ 0 & 1 \end{pmatrix}$$

Easy to verify:  $u_0 = (1, 0)^T$  is a ground state.

Orthogonal:  $u_1 = (0, 1)^T$ , but for  $v = (1, 10)^T$  we have

$$\langle J(u_1)u_1, v \rangle = 1.1 \cdot 10 = 11$$

but  $J(v) = 10$ .

$\Rightarrow u_1 = (0, 1)^T$  cannot be a singular vector.

This does not mean there is no basis of singular vectors!

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Board computation:

In what way does the proof for showing that the ground state is a singular vector fail in the case of the Rayleigh principle?



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## A basis of singular vectors?

Again, consider

$$J(u) = \|Ku\|_1 \quad \text{with} \quad K = \begin{pmatrix} 1 & -0.1 \\ 0 & 1 \end{pmatrix}$$

Then for  $q = (-1, 1)^T$  we find

$$K^T q = (-1, 1.1), KK^T q = (-1.11, 1.1)$$

such that

$$\langle K^T q, K^T q \rangle = \|K(K^T q)\|_1 = J(K^T q).$$

Moreover,

$$\langle K^T q, z \rangle = \langle q, Kz \rangle \leq \|Kz\|_1 = J(z).$$

Thus,  $K^T q \in \partial J(K^T q)$ , and after normalization,  $K^T q$  is a singular vector.



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## A basis of singular vectors?

Again, consider

$$J(u) = \|Ku\|_1 \quad \text{with} \quad K = \begin{pmatrix} 1 & -0.1 \\ 0 & 1 \end{pmatrix}$$

Then for  $q_2 = (1, 1)^T$  we find

$$K^T q_2 = (1, 0.9), KK^T q_2 = (0.91, 0.9)$$

such that

$$\langle K^T q_2, K^T q_2 \rangle = \|K(K^T q_2)\|_1 = J(K^T q_2).$$

Moreover,

$$\langle K^T q_2, z \rangle = \langle q_2, Kz \rangle \leq \|Kz\|_1 = J(z).$$

Thus,  $K^T q_2 \in \partial J(K^T q_2)$ , and after normalization,  $K^T q_2$  is a singular vector which is orthogonal to  $q$ !!



## Open problem 1: Existence of a basis

For any 1-homogeneous  $J \in \Gamma_0(\mathbb{R}^n)$  there exists a basis of generalized singular vectors?

Remark: There are some particular cases for which the above provably holds, as we will see in the next chapter.

## Open problem 2: Numerical computation of singular vectors<sup>2</sup>

The problem of finding elements that satisfy

$$\lambda v_\lambda \in \partial J(v_\lambda), \quad \|v_\lambda\|_2 = 1$$

is highly degenerate. How can we find such  $v_\lambda$  in practice?

<sup>2</sup>Also see Hein and Buehler, *An inverse power method for nonlinear eigenproblems with applications in 1-spectral clustering and sparse PCA*, 2010.

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## Optimality at zero

For any 1-homogeneous  $J \in \Gamma_0(\mathbb{R}^n)$ , if there exists no  $\alpha > 0$  such that

$$\frac{1}{\alpha} \hat{u} \in \partial J(0),$$

then  $\hat{u}$  is not a singular vector with singular value  $\lambda \neq 0$ .

## Orthogonality to kernel

For any 1-homogeneous  $J \in \Gamma_0(\mathbb{R}^n)$ , if  $v \in \text{kern}(J)$  and  $p \in \partial J(0)$  then  $p \perp v$ . In particular, all singular vectors with nonzero singular values are in  $\text{kern}(J)^\perp$ .

Conclusion: Singular vectors with singular value 0 are orthogonal to those that have non-zero singular values.

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# Error Estimates

What is the theory of nonlinear singular vectors good for?

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## Error estimates – clean data

Let  $J \in \Gamma_0(\mathbb{R}^n)$  be 1-homogeneous, and let  $v_\lambda$  be a singular vector with singular value  $\lambda$ . Then

$$u(t) = \arg \min_u \frac{1}{2} \|u - \gamma v_\lambda\|^2 + tJ(u)$$

is explicitly given by

$$u(t) = \begin{cases} (\gamma - t\lambda)v_\lambda & \text{if } t\lambda \leq \gamma, \\ 0 & \text{else.} \end{cases}$$

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Proof: Board.

## Error estimates – noisy data

Let  $J \in \Gamma_0(\mathbb{R}^n)$  be 1-homogeneous, and let  $v_\lambda$  be a singular vector with singular value  $\lambda$ . Let  $f = \gamma v_\lambda + n$ , and let there exist positive constants  $\mu, \eta$  with  $\frac{\mu}{\eta} < \gamma$  such that

$$\mu v_\lambda + \eta n \in \partial J(v_\lambda).$$

Then

$$u_\alpha = \arg \min_u \frac{1}{2} \|u - f\|^2 + \alpha J(u)$$

is explicitly given by

$$u_\alpha = \left( \gamma - \alpha \lambda + \frac{\lambda - \mu}{\eta} \right) v_\lambda,$$

if  $\alpha \in [1/\eta, 1/\eta + \gamma/\lambda[$ .

Example:  $\ell^1$  regularization – Matlab. Polyhedral discussion.



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## Remark: Generalizations

All concepts we discussed in this section can be generalized to problems of the form

$$\min_u \frac{1}{2} \|Au - f\|^2 + \alpha J(u)$$

with adapted definitions of ground states and singular vectors.  
See Benning, Burger, *Ground states and singular vectors of convex variational regularization methods.*

## Open problem 3: Arbitrary data fidelity terms

Does the theory extend to even more general variational problems like

$$\min_u H_f(u) + \alpha J(u)?$$



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# Nonlinear Multiscale Methods



Can we use variational methods like

$$u(t) = \arg \min_u \frac{1}{2} \|u - f\|^2 + tJ(u)$$

to define a multiscale decomposition of  $f$ ?

(Throughout this chapter  $J$  is 1-homogeneous.)

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For

$$u(t) = \arg \min_u \frac{1}{2} \|u - f\|^2 + tJ(u)$$

we have already seen: If  $\frac{f}{\gamma}$  is a singular vector to singular value  $\lambda$ , then

$$u(t) = \begin{cases} (1 - \frac{\lambda t}{\gamma})f & \text{if } t\lambda \leq \gamma \\ 0 & \text{else.} \end{cases}$$

How can we define a transformation that consists of a single peak if the data is a singular vector?

- $u(t)$  is piecewise linear in time.
- $\partial_t u(t)$  is piecewise constant in time.
- $\partial_{tt} u(t)$  consists of Dirac deltas (=is a measure).



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We define

$$\phi(t) = t \partial_{tt} u(t)$$

and call it the *wavelength decomposition* of  $f$  with respect to  $J$ .

- Why is it a *decomposition*?
- Why do we call it *wavelength*?

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## Spectral decomposition

It holds that

$$f - \text{proj}_{\text{kern}(J)}(f) = \int_0^\infty \phi(t) dt.$$

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For the proof we need the following statement:

## Finite time extinction

For any 1-homogeneous  $J \in \Gamma_0(\mathbb{R}^n)$  and any  $f \in \mathbb{R}^n$  there exist a  $T \in \mathbb{R}$  such that

$$u(t) = \arg \min_u \frac{1}{2} \|u - f\|_2^2 + tJ(u)$$

meets  $u(t) = \text{proj}_{\text{kern}(J)}(f)$  for all  $t \geq T$ .

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Conclusion:  $\partial_t u(t) = 0$  for all  $t \geq T$ .



## Finite time derivative

For any 1-homogeneous  $J \in \Gamma_0(\mathbb{R}^n)$  and any  $f \in \mathbb{R}^n$  we find that

$$u(t) = \arg \min_u \frac{1}{2} \|u - f\|_2^2 + tJ(u), \quad t \geq 0$$

meets  $\|\partial_t u(0)\|_2 \leq C$  for some constant  $C$ .

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We can now proof the theorem about the general decomposition.

For  $\text{proj}_{\text{kern}(J)}(f) = 0$  we have

$$f = \int_0^\infty \phi(t) dt.$$

Let us consider the example of  $\frac{f}{\gamma}$  being a singular vector again.  
We know

$$u(t) = \begin{cases} (1 - \frac{\lambda t}{\gamma})f & \text{if } t\lambda \leq \gamma \\ 0 & \text{else.} \end{cases}$$

Thus,

$$u(t) = \begin{cases} -\frac{\lambda}{\gamma}f & \text{if } t\lambda \leq \gamma \\ 0 & \text{else.} \end{cases}$$

And therefore

$$t \partial_{tt} u(t) = \frac{\lambda t}{\gamma} \delta(\lambda t - \gamma) f = \delta(\lambda t - \gamma) f$$



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