Suggested Homework 1

Nonlinear Multiscale Methods for Image and Signal Analysis

Exercise 1. Convince yourself that

- every norm is convex.
- if $E : \mathbb{R}^n \to \mathbb{R}$ is convex, then $(E \circ A)$ for $A \in \mathbb{R}^{n\times m}$ is convex.

Proof.

- Let $\|\cdot\|$ be a norm on the vector space X. Let $x, y \in X$ and $\alpha \in [0,1]$ be arbitrary. Then

$$\|\alpha x + (1-\alpha)y\| \leq \|\alpha x\| + \|(1-\alpha)y\| = \alpha\|x\| + (1-\alpha)\|y\|$$

  where we have use the triangle inequality as well as the positive homogenity of the norm.

- For $x, y \in \mathbb{R}^m$, $\alpha \in [0,1]$, we have

$$E(A(\alpha x + (1-\alpha)y)) = E(\alpha Ax + (1-\alpha)Ay) \leq \alpha E(Ax) + (1-\alpha)E(Ay),$$

  where we have used the linearity of A as well as the convexity of E.

□

Exercise 2. Prove the following (1d) statement from the lecture. If $E : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is convex, then E is locally Lipschitz on $\text{int}(\text{dom}(E))$.

Proof. Part 1: Let $x, x_1, x_2 \in \text{int}(\text{dom}(E))$ such that $x_1 < x < x_2$. Then for $\alpha = \frac{x_2 - x}{x_2 - x_1}$ we have

$$\alpha x_1 + (1-\alpha)x_2 = \alpha(x_1 - x_2) + x_2 = x.$$

We can compute

$$\begin{aligned} E(x) - E(x_1) &\leq \alpha E(x_1) + (1-\alpha)E(x_2) - E(x_1) \\ &= (1-\alpha)(E(x_2) - E(x_1)) \\ &= \frac{E(x_2) - E(x_1)}{x_2 - x_1}(x - x_1). \end{aligned}$$

On the other hand

$$\begin{aligned} E(x_2) - E(x) &\geq E(x_2) - (\alpha E(x_1) + (1-\alpha)E(x_2)) \\ &= \alpha(E(x_2) - E(x_1)) \\ &= \frac{E(x_2) - E(x_1)}{x_2 - x_1}(x_2 - x), \end{aligned}$$

such that

$$\frac{E(x) - E(x_1)}{x - x_1} \leq \frac{E(x_2) - E(x_1)}{x_2 - x_1} \leq \frac{E(x_2) - E(x)}{x_2 - x}.$$

Now for a given $x \in \operatorname{int}(\operatorname{dom}(E))$, pick $a, b, x_1, x_2 \in \operatorname{int}(\operatorname{dom}(E))$ such that $a < x_1 < x < x_2 < b$. We claim that E is Lipschitz on $]x_1, x_2[$. For any $y_1 < y_2 \in]x_1, x_2[$ we have

$$\frac{E(y_2) - E(y_1)}{y_2 - y_1} \leq \frac{E(b) - E(y_1)}{b - y_1} \leq \frac{E(b) - E(x_2)}{b - x_2} \tag{1}$$

as well as

$$\frac{E(y_2) - E(y_1)}{y_2 - y_1} \geq \frac{E(y_2) - E(a)}{y_2 - a} \geq \frac{E(x_2) - E(a)}{x_2 - a}. \tag{2}$$

Using (1) and (2) we can conclude

$$|E(y_2) - E(y_1)| \leq \max\left(\left|\frac{E(x_2) - E(a)}{x_2 - a}\right|, \left|\frac{E(b) - E(x_2)}{b - x_2}\right|\right) |y_2 - y_1|,$$

such that E is Lipschitz on $]x_1, x_2[$. □

Exercise 3. Find an example of a convex function $E : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ which is not continuous on $\operatorname{dom}(E)$.

Proof. Consider for instance

$$E(x) = \begin{cases} \infty & \text{if } x < 0, \\ 1 & \text{if } x = 0, \\ -\sqrt{x} & \text{if } x > 0. \end{cases}$$

The set $\operatorname{dom}(E) = [0, \infty[$ is convex. Furthermore

$$E(\alpha x_1 + (1-\alpha)x_2) \leq \alpha E(x_1) + (1-\alpha)E(x_2)$$

holds for $x_1, x_2 \in]0, \infty[$ since $E''(x) = \frac{1}{4x^{3/2}} > 0$ for all $x \in]0, \infty[$. The only thing we have to take care of is $x_1 = 0$. In this case

$$\begin{aligned} E(\alpha x_1 + (1-\alpha)x_2) = E((1-\alpha)x_2) &= \sqrt{(1-\alpha)x_2} \\ &\leq (1-\alpha)\sqrt{x_2} \\ &\leq \alpha + (1-\alpha)\sqrt{x_2} \\ &= \alpha E(x_1) + (1-\alpha)E(x_2). \end{aligned}$$

This shows that E is convex and obviously E is not continuous at $0 \in \operatorname{dom}(E)$. □