Suggested Homework

Nonlinear Multiscale Methods for Image and Signal Analysis

Exercise 1 (4 points). Let $E : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable convex function. Show that for all $u \in \mathbb{R}$ it holds that

$$\partial E(u) = \{E'(u)\}.$$

Proof. Let $u \in \mathbb{R}$ be arbitrary. Let $p \in \partial E(u)$. Note that we know that the subdifferential is non-empty for $u \in \mathrm{ri}(\mathrm{dom}(E)) = \mathbb{R}$. The definition of the subdifferential tells us that

$$E(v) - E(u) - p(v-u) \geq 0, \ \forall v \in \mathbb{R}.$$

Choose $v = u + \epsilon$ to see that

$$\frac{E(u+\epsilon) - E(u)}{\epsilon} \geq p,$$

and $v = u - \epsilon$ to see that

$$\frac{E(u) - E(u-\epsilon)}{\epsilon} \leq p,$$

hold for all $\epsilon > 0$. Since E is differentiable, the limit of $\epsilon \searrow 0$ exists for both left-hand-sides and is equal to the derivative. We obtain

$$p \leq E'(u), \ p \geq E'(u),$$

which yields the assertion.

$\square$