Suggested Homework

Nonlinear Multiscale Methods for Image and Signal Analysis

Exercise 1. Let $J \in \Gamma_0(\mathbb{R}^n)$ be 1-homogeneous. Prove that J^* is the indicator function of $\partial J(0)$.

Proof. By definition

$$J^*(p) = \sup_u \langle p, u\rangle - J(u).$$

Let p be arbitrary. As soon as there exists a u with

$$h := \langle p, u\rangle - J(u) > 0,$$

we find that

$$\langle p, au\rangle - J(au) = ah$$

for any $a \in \mathbb{R}^+$, such that $J^*(p) = \infty$. If there does not exist such a u, it holds that

$$J(u) \geq \langle p, u\rangle,\ \forall u, \qquad J(0) = 0 = \langle p, 0\rangle,$$

which by the characterization of the subdifferential of 1-homogeneous functions means that $p \in \partial J(0)$. In this case, the supremum in the definition of J^* is attained at $u = 0$, which yields $J^*(p) = 0$. □

Exercise 2. Let $J \in \Gamma_0(\mathbb{R}^n)$ be 1-homogeneous with $\ker(J) = \{0\}$. Show that

$$|p|_* := \sup_{u, J(u)\leq 1} \langle p, u\rangle$$

defines a norm on $\mathbb{R}^n$. Furthermore, show that $\partial J(0) = \{p \in \mathbb{R}^n \mid |p|_* \leq 1\}$.

Proof. It is clear that $|0|_* = 0$. Furthermore, for $a \in \mathbb{R}^+$ we have

$$|ap|_* = a \sup_{u, J(u)\leq 1} \langle p, u\rangle = a|p|_*.$$

For $-a \in \mathbb{R}^+$ we have

$$|ap|_* = |a| \sup_{u, J(u)\leq 1} \langle p, -u\rangle = |a| \sup_{u, J(-v)\leq 1} \langle p, v\rangle = |a| \sup_{u, J(v)\leq 1} \langle p, v\rangle = |a||p|_*.$$

Finally,

$$|p+q|_* = \sup_{u, J(u) \leq 1} (\langle p, u \rangle + \langle q, u \rangle) \leq \sup_{u, J(u) \leq 1} \langle p, u \rangle + \sup_{u, J(u) \leq 1} \langle q, u \rangle = |p|_* + |q|_*,$$

which shows that $|\cdot|_*$ is a norm.

By definition $p \in \partial J(0)$ means

$$J(v) \geq \langle p, v \rangle \ \forall v,$$

which means $|p|_* \leq 1$. $\square$

Exercise 3. Implement a function *isSubgradient = isSubgradient(u,K,q)* that verifies if $\|q\|_\infty \leq 1$ and $\langle K^T q, u \rangle = \|Ku\|_1$. Convince yourself that this implies $K^T q \in \partial J(u)$ for $J(u) = \|Ku\|_1$.

Advanced: Let K be a 1d finite difference matrix such that $J(u) = \|Ku\|_1$ is the 1d total variation. Can you construct a q such that *isSubgradient*$(q, K, K^T q)$ is true, i.e. $K^T q \in \partial J(K^T q)$?

Hint: q is piecewise linear.

The programming exercise solution is online. The advanced part will be discussed in lecture.