Suggested Homework

Nonlinear Multiscale Methods for Image and Signal Analysis

Exercise 1. Consider the gradient flow

$$\partial_t u(t) = -p(t), \qquad p(t) \in \partial J(u(t)), u(0) = f,$$

for 1-homogeneous J. Show (formally), that the definition of the spectral response as

$$S(t) = t\sqrt{\partial_{tt} J(u(t))}$$

admits a Paseval-type identity in the sense that

$$\int_0^\infty (S(t))^2 \, dt = \|f\|_2^2.$$

Hint: Integration by parts!

First of all note that the above exercise was missing that $\mathrm{proj}_{\mathrm{kern}(J)}(f) = 0$, otherwise one finds this additional quantity in the computation below.

We find that

$$\begin{aligned}
\int_0^\infty (S(t))^2 \, dt &= \int_0^\infty t^2 \partial_{tt} J(u(t)) \, dt \\
&= \underbrace{[t^2 \partial_t J(u(t)]_0^\infty}_{=0} - 2\int_0^\infty t \partial_t J(u(t)) \, dt \\
&= \underbrace{-[2tJ(u(t))]_0^\infty}_{=0} + 2\int_0^\infty J(u(t)) \, dt \\
&= 2\int_0^\infty \langle p(t), u(t)\rangle \, dt \\
&= -2\int_0^\infty \langle \partial_t u(t), u(t)\rangle \, dt \\
&= -\int_0^\infty \partial_t \|u(t)\|_2^2 \, dt \\
&= \|u(0)\|_2^2 - \underbrace{\lim_{t\to\infty} \|u(t)\|_2^2}_{=0} \\
&= \|f\|_2^2
\end{aligned}$$