Multiple View Geometry: Solution Exercise Sheet 2

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Part I: Theory

1. Groups and inclusions:
   Groups
   (a) $SO(n)$: special orthogonal group
   (b) $O(n)$: orthogonal group
   (c) $GL(n)$: general linear group
   (d) $SL(n)$: special linear group
   (e) $SE(n)$: special euclidean group (In particular, $SE(3)$ represents the rigid-body motions in $\mathbb{R}^3$)
   (f) $E(n)$: euclidean group
   (g) $A(n)$: affine group

   Inclusions
   (a) $SO(n) \subset O(n) \subset GL(n)$
   (b) $SE(n) \subset E(n) \subset A(n) \subset GL(n+1)$

2. $\lambda_a = \frac{(\lambda_a v_a)^T v_b}{\langle v_a, v_b \rangle} = \frac{v_a^T A^T v_b}{\langle v_a, v_b \rangle} = \frac{v_a^T A v_b}{\langle v_a, v_b \rangle} = \frac{v_a^T (\lambda_b v_b)}{\langle v_a, v_b \rangle} = \lambda_b$

3. Let V be the orthonormal matrix (i.e. $V^T = V^{-1}$) given by the eigenvectors, and Σ the diagonal matrix containing the eigenvalues:

$$V = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} \lambda_1 & 0 & \ddots \\ 0 & \ddots & 0 \\ \ddots & 0 & \lambda_n \end{pmatrix}.$$

As V is a basis, we can express x as a linear combination of the eigenvectors $x = V\alpha$ with $\alpha \in \mathbb{R}^n$, with $\sum_i \alpha_i^2 = \alpha^T \alpha = x^T V^T V x = x^T x = 1$. This gives

$$\begin{aligned} x^T A x &= x^T V \Sigma V^{-1} x \\ &= \alpha^T V^T V \Sigma V^T V \alpha \\ &= \alpha^T \Sigma \alpha = \sum_i \alpha_i^2 \lambda_i \end{aligned}$$

Considering $\sum_i \alpha_i^2 = 1$, we can conclude that this expression is minimized iff only the α_i corresponding to the smallest eigenvalue(s) are non-zero. If $\lambda_{n-1} \gneq \lambda_n$, there exist only two solutions ($\alpha_n = \pm 1$), otherwise infinitely many.

For maximisation, only the the α_i corresponding to the largest eigenvalue(s) can be non-zero.

4. We show that: $x \in \text{kernel}(A) \Leftrightarrow x \in \text{kernel}(A^\top A)$.

   "$\Rightarrow$": Let $x \in \text{kernel}(A)$

   $$A^\top \underbrace{Ax}_{=0} = A^T 0 = 0 \quad \Rightarrow x \in \text{kernel}(A^\top A)$$

   "$\Leftarrow$": Let $x \in \text{kernel}(A^T A)$

   $$0 = x^T \underbrace{A^T Ax}_{=0} = \langle Ax, Ax \rangle = ||Ax||^2 \quad \Rightarrow Ax = 0 \quad \Rightarrow x \in \text{kernel}(A)$$

5. Singular Value Decomposition (SVD)

   (a) $A \in \mathbb{R}^{m \times n}$ with $m \geq n$, $U \in \mathbb{R}^{m \times m}$, $S \in \mathbb{R}^{m \times n}$, $V \in \mathbb{R}^{n \times n}$

   (b) Similarities and differences between SVD and EVD:

      i. Both are matrix diagonalization techniques.

      ii. The SVD can be applied to matrices $A \in \mathbb{R}^{m \times n}$ with $m \neq n$, whereas the EVD is only applicable to quadratic matrices ($A \in \mathbb{R}^{m \times n}$ with $m = n$).

   (c) Relationship between U, S, V and the eigenvalues and eigenvectors of $A^\top A$ and $AA^\top$:

      i. the columns of V are eigenvectors of $A^\top A$

      ii. the columns of U are eigenvectors of $AA^\top$

   (d) Entries in S:

      i. S is a diagonal matrix. The elements along the diagonal are the *singular values* of A.

      ii. The number of non-zero singular values gives us the *rank* of the matrix A.