



Multiple View Geometry: Solution Exercise Sheet 3

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<http://vision.in.tum.de/teaching/ss2015/mvg2015>

Part I: Theory

$$1. \quad (a) \quad M = \begin{pmatrix} I & T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(b) \quad M = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(c) \quad M = \begin{pmatrix} I & T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(d) \quad M = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I & T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R & RT \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & r_1 t_x \\ r_{21} & r_{22} & r_{23} & r_2 t_y \\ r_{31} & r_{32} & r_{33} & r_3 t_z \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where r_1, r_2, r_3 are the row vectors of R : $R = \begin{pmatrix} -r_1 & - \\ -r_2 & - \\ -r_3 & - \end{pmatrix}$.

$$2. \quad \text{Let } M := (M_1 - M_2) =: \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}.$$

” \Rightarrow ”:

We show that M is skew-symmetric by distinguishing diagonal and off-diagonal elements of M :

$$(a) \quad \forall i: 0 = e_i^T M e_i = m_{ii}$$

where e_i = i -th unit vector

$$(b) \quad \forall i \neq j: 0 = (e_i + e_j)^T M (e_i + e_j)$$

where e_j = j -th unit vector

$$= m_{ii} + m_{jj} + m_{ij} + m_{ji} \Rightarrow m_{ij} = -m_{ji}$$

hence, $m_{ii} = 0$ and $m_{ij} = -m_{ji}$, i.e. M is skew-symmetric.

” \Leftarrow ”:

using $M = -M^T$, we directly calculate

$$\forall x: x^T M x = (x^T M x)^T = x^T M^T x = -(x^T M x)$$

$$\Rightarrow x^T M x = 0$$

3. We know: $\omega = (\omega_1 \ \omega_2 \ \omega_3)^T$ with $\|\omega\| = 1$ and $\hat{\omega} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$.

(a)

$$\begin{aligned}
\hat{\omega}^2 &= \begin{pmatrix} -(\omega_2^2 + \omega_3^2) & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & -(\omega_1^2 + \omega_3^2) & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & -(\omega_1^2 + \omega_2^2) \end{pmatrix} \\
&= \begin{pmatrix} \omega_1^2 - \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2)}_1 & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & \omega_2^2 - \underbrace{(\omega_2^2 + \omega_1^2 + \omega_3^2)}_1 & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & \omega_3^2 - \underbrace{(\omega_3^2 + \omega_1^2 + \omega_2^2)}_1 \end{pmatrix} \\
&= \begin{pmatrix} \omega_1^2 & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & \omega_2^2 & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & \omega_3^2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \omega\omega^T - \mathbf{I}
\end{aligned}$$

$$\begin{aligned}
\hat{\omega}^3 &= \hat{\omega} \hat{\omega}^2 \\
&= \hat{\omega} (\omega\omega^T - \mathbf{I}) \\
&= \hat{\omega} \omega (\omega^T) - \hat{\omega} \mathbf{I} \\
&= (\omega \times \omega) \omega^T - \hat{\omega} \\
&= -\hat{\omega} \quad (\text{as } \omega \times \omega = 0)
\end{aligned}$$

Alternative solution for $\hat{\omega}^3$:

$$\begin{aligned}
\hat{\omega}^3 &= \begin{pmatrix} -(\omega_2^2 + \omega_3^2) & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & -(\omega_1^2 + \omega_3^2) & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & -(\omega_1^2 + \omega_2^2) \end{pmatrix} \cdot \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \omega_3 \cdot \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2)}_1 & -\omega_2 \cdot \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2)}_1 \\ -\omega_3 \cdot \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2)}_1 & 0 & \omega_1 \cdot \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2)}_1 \\ \omega_2 \cdot \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2)}_1 & -\omega_1 \cdot \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2)}_1 & 0 \end{pmatrix} \\
&= -\hat{\omega}
\end{aligned}$$

(b) The formulas for n even and odd can be found by writing down the solutions for $n = 1, \dots, 6$:

$$\begin{array}{ll}
 \hat{\omega} & \\
 \hat{\omega}^2 & \\
 \hat{\omega}^3 = -\hat{\omega} & \\
 \hat{\omega}^4 = -\hat{\omega}^2 & \text{as: } \hat{\omega}^4 = \hat{\omega}^3 \hat{\omega} = -\hat{\omega} \hat{\omega} = -\hat{\omega}^2 \\
 \hat{\omega}^5 = \hat{\omega} & \text{as: } \hat{\omega}^5 = \hat{\omega}^4 \hat{\omega} = -\hat{\omega}^2 \hat{\omega} = -\hat{\omega}^3 = -(-\hat{\omega}) = \hat{\omega} \\
 \hat{\omega}^6 = \hat{\omega}^2 & \text{as: } \hat{\omega}^6 = \hat{\omega}^5 \hat{\omega} = \hat{\omega} \hat{\omega} = \hat{\omega}^2
 \end{array}$$

For even numbers:

$$\begin{array}{l}
 \hat{\omega}^2 \\
 \hat{\omega}^4 = -\hat{\omega}^2 \\
 \hat{\omega}^6 = \hat{\omega}^2
 \end{array}$$

For odd numbers:

$$\begin{array}{l}
 \hat{\omega} \\
 \hat{\omega}^3 = -\hat{\omega} \\
 \hat{\omega}^5 = \hat{\omega}
 \end{array}$$

$$\begin{array}{l}
 n \text{ even: } \hat{\omega}^n = (-1)^{\frac{n}{2}+1} \hat{\omega}^2 \\
 n \text{ odd: } \hat{\omega}^n = (-1)^{\frac{n-1}{2}} \hat{\omega}
 \end{array}$$

Proof via complete induction:

i. For even numbers n :

- $n = 2$: $\hat{\omega}^2 = (-1)^{\frac{2}{2}+1} \hat{\omega}^2$
- Induction step $n \rightarrow n + 2$:

$$\begin{aligned}
 \hat{\omega}^{n+2} &= \hat{\omega}^n \cdot \hat{\omega}^2 \\
 &= (-1)^{\frac{n}{2}+1} \cdot \hat{\omega}^2 \cdot \hat{\omega}^2 && \text{(assumption)} \\
 &= (-1)^{\frac{n}{2}+1} \cdot \hat{\omega}^3 \cdot \hat{\omega} \\
 &\stackrel{(a)}{=} (-1)^{\frac{(n+2)}{2}+1} \cdot \hat{\omega}^2
 \end{aligned}$$

ii. For odd numbers n :

- $n = 3$: $\hat{\omega}^3 = -\hat{\omega} = (-1)^{\frac{3-1}{2}} \hat{\omega}$
- Induction step $n \rightarrow n + 2$:

$$\begin{aligned}
 \hat{\omega}^{n+2} &= \hat{\omega}^n \cdot \hat{\omega}^2 \\
 &= (-1)^{\frac{n-1}{2}} \cdot \hat{\omega} \cdot \hat{\omega}^2 && \text{(assumption)} \\
 &= (-1)^{\frac{n-1}{2}} \cdot \hat{\omega}^3 \\
 &\stackrel{(a)}{=} (-1)^{\frac{n-1}{2}+1} \cdot \hat{\omega} \\
 &= (-1)^{\frac{(n+2)-1}{2}} \cdot \hat{\omega}
 \end{aligned}$$

(c) We know: $\omega \in \mathbb{R}^3$. Let $v = \frac{\hat{\omega}}{\|\omega\|}$ and $t = \|\omega\|$. Hence, $w = vt$.

$$\begin{aligned}
 e^{\hat{\omega}} &= e^{\hat{v}t} \\
 &= \sum_{n=0}^{\infty} \frac{(\hat{v}t)^n}{n!} \\
 &\stackrel{(b)}{=} I + \underbrace{\sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^{2n}}{(2n)!}}_{1 - \cos(t)} \hat{v}^2 + \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!}}_{\sin(t)} \hat{v} \\
 &\stackrel{(\text{def.})}{=} I + \frac{\hat{\omega}^2}{\|\omega\|^2} (1 - \cos(\|\omega\|)) + \frac{\hat{\omega}}{\|\omega\|} \sin(\|\omega\|)
 \end{aligned}$$