Existence and uniqueness of the projection

Let C be a nonempty closed convex set. Then there exists one unique element in $\pi_C(v)$.

Proof. We define

$$E(u) = \begin{cases} \|u - v\|_2^2 & \text{if } u \in C, \\ \infty & \text{else,} \end{cases}$$

Let $v_0 \in C$. Then $S_{E(v_0)} = \{u \mid E(u) \leq E(v_0)\}$ is nonempty and bounded. A short computation shows that the closedness of C implies the closedness of $\operatorname{epi}(E)$, and thus yields the existence of a minimizer. Since E is strictly convex, the minimizer is unique. □

Convergence Analysis of the Gradient Projection Algorithm

Let us first consider what we know about a projection. We have

$$u^{k+1} \in \operatorname*{argmin}_u \frac{1}{2}\|u - u^k + \tau \nabla E(u^k)\|^2 + \iota_C(u) \tag{1}$$

where

$$\iota_C(u) = \begin{cases} 0 & \text{if } u \in C, \\ \infty & \text{otherwise.} \end{cases}$$

The optimality condition to (1) tells us that

$$\begin{aligned} 0 &= u^{k+1} - u^k + \tau \nabla E(u^k) + p^{k+1} \qquad p^{k+1} \in \partial \iota_C(u^{k+1}), \\ \Rightarrow & u^k - u^{k+1} - \tau \nabla E(u^k) \in \partial \iota_C(u^{k+1}). \end{aligned}$$

By definition of the subdifferential we know that this means

$$\langle u^k - u^{k+1} - \tau \nabla E(u^k), u^{k+1} - u \rangle \geq 0 \qquad \forall u \in C,$$

which means that

$$\frac{1}{\tau}\langle u^k - u^{k+1}, u^{k+1} - u \rangle \geq \langle \nabla E(u^k), u^{k+1} - u \rangle \qquad \forall u \in C, \tag{2}$$

could be a useful estimate.

Now let us consider what we know about our problem for $u \in C$, $u^k \in \mathbb{R}^n$, and $u^{k+1} = \pi_C(u^k - \tau \nabla E(u^k))$. The m-strong convexity and L-smoothness of E tell us that

$$\begin{aligned} 0 &\leq E(u) - E(u^k) - \langle \nabla E(u^k), u - u^k \rangle - \frac{m}{2}\|u - u^k\|^2, \\ 0 &\leq -\Big(E(u^{k+1}) - E(u^k) - \langle \nabla E(u^k), u^{k+1} - u^k \rangle - \frac{L}{2}\|u^{k+1} - u^k\|^2\Big). \end{aligned}$$

Adding the two inequalities with $\tau = \frac{1}{L}$ yields

$$
\begin{aligned}
0 &\leq E(u) - E(u^{k+1}) + \langle \nabla E(u^k), u^{k+1} - u \rangle - \frac{m}{2}\|u - u^k\|^2 + \frac{L}{2}\|u^{k+1} - u^k\|^2 \\
&\overset{(2)}{\leq} E(u) - E(u^{k+1}) + L\langle u^k - u^{k+1}, u^{k+1} - u \rangle - \frac{m}{2}\|u - u^k\|^2 + \frac{L}{2}\|u^{k+1} - u^k\|^2 \\
&= E(u) - E(u^{k+1}) + \frac{L}{2}\|u^{k+1} - u^k\|^2 + L\langle u^k - u^{k+1}, u^{k+1} - u \rangle + \frac{L}{2}\|u^{k+1} - u\|^2 \\
&- \frac{L}{2}\|u^{k+1} - u\|^2 - \frac{m}{2}\|u - u^k\|^2 \\
&= E(u) - E(u^{k+1}) + \frac{L}{2}\|u^k - u\|^2 - \frac{L}{2}\|u^{k+1} - u\|^2 - \frac{m}{2}\|u - u^k\|^2 \\
&= E(u) - E(u^{k+1}) - \frac{L}{2}\|u^{k+1} - u\|^2 + \frac{L - m}{2}\|u - u^k\|^2
\end{aligned}
$$