Chapter 1
Convex Analysis

Convex Optimization for Computer Vision
SS 2016

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Convexity

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Convex energy minimization problems

This lecture is all about

$$\hat{u} = \arg\min_{u \in C} E(u),$$

where $C \subset \mathbb{R}^n$ convex set, $E : C \to \mathbb{R}$ convex function.

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1. What is a convex set?

Definition

A set $C \subset \mathbb{R}^n$ is called convex, if

$$\alpha x + (1-\alpha) y \in C, \quad \forall x, y \in C,\ \forall \alpha \in [0,1].$$

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$\rightarrow$ *Draw a picture.*

$\rightarrow$ *Online TED.*

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Convex set - summary of online TED

Definitions and things to recall from analysis 1:

Definitions

- A set $C \subset \mathbb{R}^n$ is called **open** if for all $x \in C$ there exists a $\epsilon > 0$ such that the ball with radius ϵ around x, $B(x, \epsilon)$, is contained in C: $B(x, \epsilon) \subset C$.

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- A set is closed if and only if it contains all its limit points.
- The **closure** of a set $C \subset \mathbb{R}^n$ is

$$\overline{C} = \{x \mid \text{ there exists a convergent sequence } (x_n)_n \subset C \text{ such that } \lim_{n \to \infty} x_n = x\}$$

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- The **interior** of a set $C \subset \mathbb{R}^n$ is

$$\mathring{C} = \{x \in C \mid \text{ there exists } \epsilon > 0 \text{ such that } B(x, \epsilon) \subset C\}$$

Convex set - summary of online TED

The following operations preserve the convexity of a set

- Intersection
- Vector sum
- Closure
- Interior
- Linear Transformation

The union of convex sets is not convex in general.

Polyhedral sets are always convex, cones are not necessarily convex.

Convex energy minimization problems

Let's get back to what the lecture is all about:

$$\hat{u} = \arg\min_{u \in C} E(u),$$

where $C \subset \mathbb{R}^n$ convex set, $E : C \to \mathbb{R}$ convex function.

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1. What is a convex set? We know this now!

2. What is a convex function?

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1. What is a convex set? We know this now!

2. What is a convex function?

Definition: Convex Function

We call $E : C \to \mathbb{R}$ a convex function if C is a convex set and for all $u, v \in C$ and all $\theta \in [0, 1]$ it holds that

$$E(\theta u + (1-\theta)v) \leq \theta E(u) + (1-\theta)E(v)$$

We call E strictly convex, if the inequality is strict for all $\theta \in]0, 1[$, and $v \neq u$.

$\to$ *Draw a picture.*

Convex functions - summary of online TED

The following operations do preserve the convexity of a function

- Summation
- Linear Transformation

Convex functions - summary of online TED

The following operations do preserve the convexity of a function

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- Linear Transformation

The following operations do not preserve the convexity of a function

- Multiplication, Division, Difference
- Composition

Convex functions - summary of online TED

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The sum of a convex function and a strictly convex function is strictly convex.

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- Summation
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The following operations do not preserve the convexity of a function

- Multiplication, Division, Difference
- Composition

The sum of a convex function and a strictly convex function is strictly convex.

Remember to check two conditions to show that a function is convex!

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Convex energy minimization problems

Let's get back to what the lecture is all about:

$$\hat{u} = \arg\min_{u \in C} E(u), \qquad (1)$$

where $C \subset \mathbb{R}^n$ convex set, $E : C \to \mathbb{R}$ convex function.

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Let's get back to what the lecture is all about:

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where $C \subset \mathbb{R}^n$ convex set, $E : C \to \mathbb{R}$ convex function.

It is sometimes convenient to "introduce" the constraint $u \in C$ into the energy function E itself. We therefore introduce the notion of **extended real valued functions**.

$$E : \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}.$$

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The minimization problem (1) can then be written as

$$\hat{u} = \arg\min_{u \in \mathbb{R}^n} \tilde{E}(u),$$

by defining

$$\tilde{E}(u) = \begin{cases} E(u) & \text{if } u \in C, \\ \infty & \text{else.} \end{cases}$$

Extended real valued functions

Definition

- For $E : \mathbb{R}^n \to \overline{\mathbb{R}}$, we call

$$\mathrm{dom}(E) := \{u \in \mathbb{R}^n \mid E(u) < \infty\}$$

  the domain of E.
- We call E proper if $\mathrm{dom}(E) \neq \emptyset$.

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- We call E proper if $\text{dom}(E) \neq \emptyset$.

Revisiting the definition of convex functions

We call $E : \mathbb{R}^n \to \overline{\mathbb{R}}$ a convex function if

1. $\text{dom}(E)$ is a convex set.
2. For all $u, v \in \text{dom}(E)$ and all $\theta \in [0, 1]$ it holds that

$$E(\theta u + (1-\theta)v) \leq \theta E(u) + (1-\theta)E(v)$$

We call E strictly convex, if the inequality in 2 is strict for all $\theta \in]0, 1[$, and $v \neq u$.

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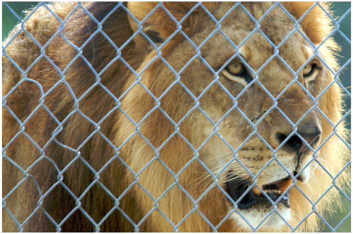
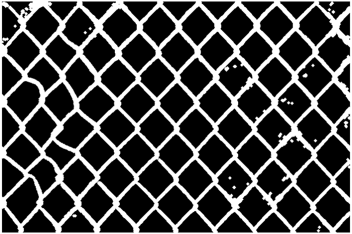
First example of an imaging problem: Inpainting

Example: Inpainting





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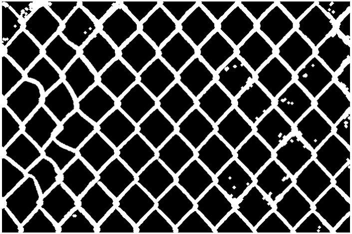

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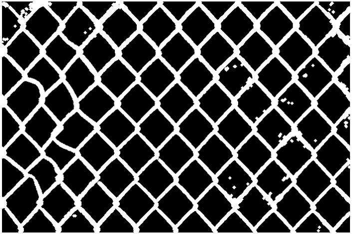

$$\min_{u \in \mathbb{R}^{n \times m}} \sum_{i,j} (u_{i,j} - u_{i-1,j})^2 + (u_{i,j} - u_{i,j-1})^2 \quad \text{s.t. } u_{i,j} = f_{i,j} \; \forall (i,j) \in I$$

with index set I of pixels to keep and suitable boundary conditions.

→ *Discuss convexity.*

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