Chapter 1
Convex Analysis

Convex Optimization for Computer Vision
SS 2016

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Convexity

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Convex energy minimization problems

This lecture is all about

$$\hat{u} \in \arg\min_{u \in C} E(u),$$

where $C \subset \mathbb{R}^n$ convex set, $E : C \to \mathbb{R}$ convex function.

1. What is a convex set?

Definition

A set $C \subset \mathbb{R}^n$ is called convex, if

$$\alpha x + (1-\alpha) y \in C, \quad \forall x, y \in C, \ \forall \alpha \in [0,1].$$

$\rightarrow$ *Draw a picture.*

$\rightarrow$ *Online TED.*

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Convex set - summary of online TED

Definitions and things to recall from analysis 1:

Definitions

- A set $C \subset \mathbb{R}^n$ is called **open** if for all $x \in C$ there exists a $\epsilon > 0$ such that the ball with radius ϵ around x, $B(x, \epsilon)$, is contained in C: $B(x, \epsilon) \subset C$.
- A set $C \subset \mathbb{R}^n$ is called **closed** if its complement is open.
- A set is closed if and only if it contains all its limit points.
- The **closure** of a set $C \subset \mathbb{R}^n$ is

$$\overline{C} = \{x \mid \text{ there exists a convergent sequence } (x_n)_n \subset C \text{ such that } \lim_{n\to\infty} x_n = x\}$$

- The **interior** of a set $C \subset \mathbb{R}^n$ is

$$\mathring{C} = \{x \in C \mid \text{ there exists } \epsilon > 0 \text{ such that } B(x, \epsilon) \subset C\}$$

Convex set - summary of online TED

The following operations preserve the convexity of a set

- Intersection
- Vector sum
- Closure
- Interior
- Linear Transformation

The union of convex sets is not convex in general.

Polyhedral sets are always convex, cones are not necessarily convex.

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Convex energy minimization problems

Let's get back to what the lecture is all about:

$$\hat{u} \in \arg\min_{u \in C} E(u),$$

where $C \subset \mathbb{R}^n$ convex set, $E : C \to \mathbb{R}$ convex function.

1. What is a convex set? We know this now!

2. What is a convex function?

Definition: Convex Function

We call $E : C \to \mathbb{R}$ a convex function if C is a convex set and for all $u, v \in C$ and all $\theta \in [0, 1]$ it holds that

$$E(\theta u + (1 - \theta)v) \leq \theta E(u) + (1 - \theta)E(v)$$

We call E strictly convex, if the inequality is strict for all $\theta \in]0, 1[$, and $v \neq u$.

$\rightarrow$ *Draw a picture.*

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Convex functions - summary of online TED

The following operations do preserve the convexity of a function

- Summation
- Linear Transformation

The following operations do not preserve the convexity of a function

- Multiplication, Division, Difference
- Composition

The sum of a convex function and a strictly convex function is strictly convex.

Remember to check two conditions to show that a function is convex!

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Convex energy minimization problems

Let's get back to what the lecture is all about:

$$\hat{u} \in \arg\min_{u \in C} E(u), \tag{1}$$

where $C \subset \mathbb{R}^n$ convex set, $E : C \to \mathbb{R}$ convex function.

It is sometimes convenient to "introduce" the constraint $u \in C$ into the energy function E itself. We therefore introduce the notion of **extended real valued functions**.

$$E : \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}.$$

The minimization problem (1) can then be written as

$$\hat{u} \in \arg\min_{u \in \mathbb{R}^n} \tilde{E}(u),$$

by defining

$$\tilde{E}(u) = \begin{cases} E(u) & \text{if } u \in C, \\ \infty & \text{else.} \end{cases}$$

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Extended real valued functions

Definition

- For $E : \mathbb{R}^n \to \overline{\mathbb{R}}$, we call

$$\operatorname{dom}(E) := \{u \in \mathbb{R}^n \mid E(u) < \infty\}$$

  the domain of E.
- We call E proper if $\operatorname{dom}(E) \neq \emptyset$.

Revisiting the definition of convex functions

We call $E : \mathbb{R}^n \to \overline{\mathbb{R}}$ a convex function if

1. $\operatorname{dom}(E)$ is a convex set.

2. For all $u, v \in \operatorname{dom}(E)$ and all $\theta \in [0, 1]$ it holds that

$$E(\theta u + (1-\theta)v) \leq \theta E(u) + (1-\theta)E(v)$$

We call E strictly convex, if the inequality in 2 is strict for all $\theta \in]0, 1[$, and $v \neq u$.

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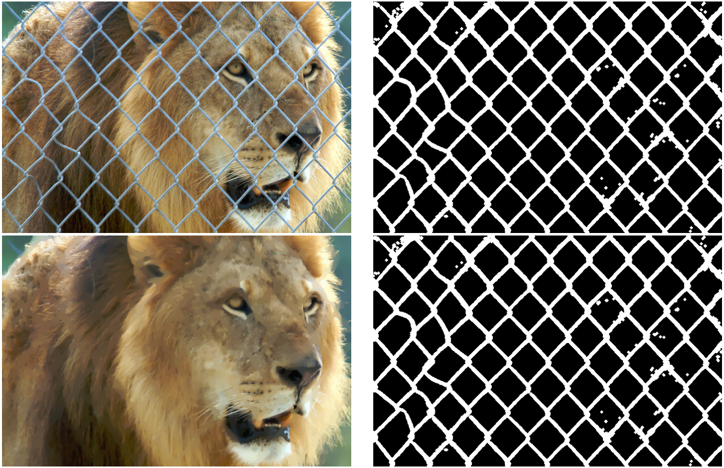
First example of an imaging problem: Inpainting

Example: Inpainting



$$\min_{u \in \mathbb{R}^{n \times m}} \sum_{i,j} (u_{i,j} - u_{i-1,j})^2 + (u_{i,j} - u_{i,j-1})^2 \quad \text{s.t. } u_{i,j} = f_{i,j} \; \forall (i,j) \in I$$

with index set I of pixels to keep and suitable boundary

Epigraph of a function

Is there a connection between convex sets and functions?

Defintion: Epigraph

Let $E : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper function mapping into the extended real line. Then

$$\text{epi}(E) := \{(u, \alpha) \mid E(u) \leq \alpha\}$$

is called the *epigraph* of the function E.

$\rightarrow$ *Draw a picture.*

Theorem

A proper function $E : \mathbb{R}^n \to \overline{\mathbb{R}}$ is convex if and only if it's epigraph is convex

Proof: First exercise sheet.

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Properties of convex functions

What is so special about convex functions?

Theorem

Let $E : \mathbb{R}^n \to \overline{\mathbb{R}}$ be convex. Any local minimum of E is global.

Proof: Board.

Theorem: Monotonicity of the gradient

Let $E : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper, convex and differentiable at $u \in \mathrm{dom}(E)$.

$$E(v) - E(u) - \langle \nabla E(u), v - u \rangle \geq 0 \qquad \forall v \in \mathbb{R}^n$$

Proof: Later.

Conclusion

Let $E : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper, convex and differentiable at $u \in \mathrm{dom}(E)$. If $\nabla E(u) = 0$ then u is a global minimum of E.

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Convex energy minimization problems

We said this lecture is all about

$$\hat{u} \in \arg\min_{u \in \mathbb{R}^n} E(u),$$

where $E : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a convex function.

Does a minimizer of such a function even exist?

Existence of minimizers

Defintion: Lower semi-continuity (l.s.c.)

We call the function $E : \mathbb{R}^n \to \overline{\mathbb{R}}$ lower semi-continuous (l.s.c.), if for all u it holds that

$$\liminf_{v \to u} E(v) \geq E(u).$$

Theorem: Existence of minimizers

Let $E : \mathbb{R}^n \to \overline{\mathbb{R}}$ be l.s.c. and let there exist an α such that the sublevelset

$$\{u \in \mathbb{R}^n \mid E(u) \leq \alpha\}$$

is nonempty and bounded, then

$$\hat{u} \in \arg\min_u E(u)$$

exists.

Proof: Board.

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Closedness and lower semi-continuity

Defintion: Closed function

We call the function $E : \mathbb{R}^n \to \overline{\mathbb{R}}$ closed if it's epigraph is closed.

Theorem: Equivalence of l.s.c. and closedness

For $E : \mathbb{R}^n \to \overline{\mathbb{R}}$ the following two statements are equivalent

- E is lower semi-continuous (l.s.c.).
- E is closed.

Proof: Board.

OnlineTED!

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Continuity of convex functions

Continuity of Convex Functions

If $E : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is convex, then E is locally Lipschitz (and hence continuous) on int(dom(E)).

Proof in 1d: Exercise for yourself (solution will be online)

$\to$ *Board: Considering the interior is important!*

Conclusion

If $E : \mathbb{R}^n \to \mathbb{R}$ is convex, then E is continuous.

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Existence of minimizers part II

Definition: Coercivity

A function $E : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is called *coercive* if $E(v_n) \to \infty$ for all sequences $(v_n)_n$ with $\|v_n\| \to \infty$.

Remark: Coercivity implies that there exists a bounded sublevelset.

Existence of a minimizer for function with full domain

Let $E : \mathbb{R}^n \to \mathbb{R}$ be convex and coercive, then an element $\hat{u} \in \arg\min_u E(u)$ exists.

Proof:

- $\text{dom}(E) = \mathbb{R}^n$, E convex $\Rightarrow$ E is continuous.
- E is coercive, i.e. there exists a non-empty bounded sublevelset.

Uniqueness

When is

$$\hat{u} \in \arg\min_{u \in \mathbb{R}^n} E(u)$$

unique?

Theorem: Uniqueness

If $E : \mathbb{R}^n \to \overline{\mathbb{R}}$ is strictly convex, then there exists at most one local minimum which is the unique global minimum.

Proof: Simple computation.

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Optimality conditions

How can we determine if

$$\hat{u} \in \arg\min_{u \in \mathbb{R}^n} E(u)? \tag{1}$$

In other words,

what is the optimality condition for (1)?

Consider a differentiable E and remember analysis I:

Necessary condition for local extremum is

$$\nabla E(\hat{u}) = 0$$

Sufficient condition? **Convexity!**

Optimality conditions

Example why convex functions are great

(stealing from Thomas' examples)

$$\min_u \frac{1}{2}\|u - f\|_2^2 + \alpha R(\nabla u)$$

with

$$R(d) = \begin{cases} \frac{1}{2}\|d\|_2^2 & \text{if } \|d\|_2 \leq \epsilon \\ \epsilon\|d\| - \frac{1}{2}\epsilon^2 & \text{else.} \end{cases}$$

Examples: Derivatives of convex functions

Getting back: What are the optimality conditions for ...

... $E(u) = \|u - f\|_2^2 = \sum_{i=1}^n (u_i - f_i)^2$?

... $E(u) = \|Au - f\|_2^2$ for a matrix $A \in \mathbb{R}^{m \times n}$?

... $E(u) = \|u\|_1 = \sum_{i=1}^n |u_i|$?

We need a theory for non-differentiable functions!

Illustrate ℓ^1 case for discussion.

The subdifferential

Definition: Subdifferential

Let $E : \mathbb{R}^n \to \overline{\mathbb{R}}$ be convex. We call

$$\partial E(u) = \{p \in \mathbb{R}^n \mid E(v) - E(u) - \langle p, v - u \rangle \geq 0,\ \forall v \in \mathbb{R}^n\}$$

the subdifferential of E at u.

- Elements of $\partial E(u)$ are called subgradients.
- If $\partial E(u) \neq \emptyset$, we call E subdifferentiable at u.
- By convention, $\partial E(u) = \emptyset$ for $u \neq \mathrm{dom}(E)$.

Theorem: Optimality condition

Let $0 \in \partial E(\hat{u})$. Then $\hat{u} \in \arg\min_u E(u)$.

The subdifferential

Examples for non-differentiable functions:

- The absolute value function.
- Functional

$$E(u) = \begin{cases} 0 & \text{if } u \geq 0 \\ \infty & \text{else.} \end{cases}$$

- $E(u) = \frac{1}{2}\|u\|^2$

Subdifferential and derivatives

Let the convex function $E : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be differentiable at $u \in \text{int}(\text{dom}(E))$. Then

$$\partial E(u) = \{\nabla E(u)\}.$$

Proof: Exercise.

This also proves the "Theorem: Monotonicity of the gradient".

The subdifferential

Geometric interpretation of subgradients:

Any subgradient $p \in \partial E(u)$ represents a non-vertical supporting hyperplane to $\text{epi}(E)$ at $(u, E(u))$.

Definition

A supporting hyperplane to a set $S \subset \mathbb{R}^n$ is a hyperplane $\{x \in \mathbb{R}^n \mid \langle a, x \rangle = b\}$, such that

- $S \subset \{x \in \mathbb{R}^n \mid \langle a, x \rangle \leq b\}$ or $S \subset \{x \in \mathbb{R}^n \mid \langle a, x \rangle \geq b\}$
- $\exists y \in \partial S$ (the boundary of S) such that $\langle a, y \rangle = b$.

Let $p \in \partial E(u)$. Then

$$\begin{aligned} & E(v) - E(u) - \langle p, v - u \rangle \geq 0 && \forall v \in \mathbb{R}^n \\ \Rightarrow \quad & \alpha - E(u) - \langle p, v - u \rangle \geq 0 && \forall (v, \alpha) \in \text{epi}(E) \\ \Rightarrow \quad & \left\langle \begin{bmatrix} -p \\ 1 \end{bmatrix}, \begin{bmatrix} v \\ \alpha \end{bmatrix} - \begin{bmatrix} u \\ E(u) \end{bmatrix} \right\rangle \geq 0 && \forall (v, \alpha) \in \text{epi}(E). \end{aligned}$$

$\rightarrow$ *Draw image on the board.*

The subdifferential

Is any convex E subdifferentiable at $x \in \mathrm{dom}(E)$?

$$E(u) = \begin{cases} -\sqrt{u} & \text{if } u \geq 0 \\ \infty & \text{else.} \end{cases}$$

Definition: Relative Interior

The *relative interior* of a convex set M is defined as

$$\mathrm{ri}(M) := \{x \in M \mid \forall y \in M,\ \exists \lambda > 1,\ \text{s.t. } \lambda x + (1-\lambda) y \in M\}$$

Theorem: Subdifferentiability[1]

If E is a proper convex function and $u \in \mathrm{ri}(\mathrm{dom}(E))$, then $\partial E(u)$ is non-empty and bounded.

Partial proof on the board, full proof Rockafellar.

[1] Rockafellar, Convex Analysis, Theorem 23.4

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Subdifferential rules

Theorem: Sum rule[2]

Let E_1, E_2 be convex functions such that

$$\mathrm{ri}(\mathrm{dom}(E_1)) \cap \mathrm{ri}(\mathrm{dom}(E_2)) \neq \emptyset,$$

then it holds that

$$\partial(E_1 + E_2)(u) = \partial E_1(u) + \partial E_2(u).$$

Example: Minimize $(u - f)^2 + \iota_{u \geq 0}(u)$.

Example: Minimize $0.5(u - f)^2 + \alpha|u|$.

[2] Rockafellar, Convex Analysis, Theorem 23.8

Subdifferential rules

Theorem: Chain rule[3]

If $A \in \mathbb{R}^{m \times n}$, $E : \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ is convex, and $\text{ri}(\text{dom}(E)) \cap \text{range}(A) \neq \emptyset$, then

$$\partial(E \circ A)(u) = A^* \partial E(Au)$$

Example: Minimize $\|Au - f\|_2^2$.

[3] Rockafellar, Convex Analysis, Theorem 23.9

Subdifferential rules

We have seen the example of ℓ^1 minimization/denoising

$$\min_u \frac{1}{2}\|u - f\|_2^2 + \alpha\|u\|_1$$

More interesting for imaging: Change of basis, e.g. orthogonal wavelet basis

$$\min_u \frac{1}{2}\|u - f\|_2^2 + \alpha\|Wu\|_1$$

Show example in Matlab!

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Summary

- **Convex functions**
  - Every local minimum is global
  - First order optimality condition is sufficient
- The **optimality condition** for $\hat{u}$ to minimize E is

$$0 \in \partial E(\hat{u})$$

- **The subdifferential** $\partial E(u)$
  - is set valued.
  - generalizes the derivative.
  - $\partial E(u) = \{\nabla E(u)\}$ is E is differentiable at u.
  - can be identified with supporting hyperplanes to epi(E).
  - Obeys the “usual” sum and chain rules.

We now have all tools that are necessary to discuss a first class of minimization algorithms for determining

$$\hat{u} \in \underset{u}{\operatorname{argmin}}\, E(u)$$

Next lecture: Implementation, convergence, applications!

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