

Infimal convolution and its convex conjugate

The infimal convolution $f \square g$ of two functions $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ and $g : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ is defined as

$$(f \square g)(u) = \inf_{v \in \mathbb{R}^n} f(u - v) + g(v)$$

Let f, g be proper on \mathbb{R}^n . Then $(f \square g)^*(v) = f^*(v) + g^*(v)$.

Proof.

$$\begin{aligned} (f \square g)^*(v) &= \sup_{u \in \mathbb{R}^n} \langle u, v \rangle - (f \square g)(u) \\ &= \sup_{u \in \mathbb{R}^n} \langle u, v \rangle - \inf_{p \in \mathbb{R}^n} f(u - p) + g(p) \\ &= \sup_{u \in \mathbb{R}^n} \langle u, v \rangle + \sup_{p \in \mathbb{R}^n} -f(u - p) - g(p) \\ &= \sup_{\substack{u \in \mathbb{R}^n, \\ p \in \mathbb{R}^n}} \langle u, v \rangle - f(u - p) - g(p) \end{aligned}$$

We introduce the substitution $z := u - p \iff u = z + p$, eliminate u and obtain:

$$\begin{aligned} \dots &= \sup_{\substack{z \in \mathbb{R}^n, \\ p \in \mathbb{R}^n}} \langle z + p, v \rangle - f(z) - g(p) \\ &= \sup_{\substack{z \in \mathbb{R}^n, \\ p \in \mathbb{R}^n}} \langle z, v \rangle - f(z) + \langle p, v \rangle - g(p) \\ &= \sup_{z \in \mathbb{R}^n} \langle z, v \rangle - f(z) + \sup_{p \in \mathbb{R}^n} \langle p, v \rangle - g(p) \\ &= f^*(v) + g^*(v) \end{aligned}$$

□

The Huber penalty and its convex conjugate

The Huber penalty $h_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$ is given as

$$h_\varepsilon(x) = \begin{cases} \frac{x^2}{2\varepsilon} & \text{if } |x| \leq \varepsilon, \\ |x| - \frac{\varepsilon}{2} & \text{otherwise.} \end{cases}$$

It holds that

$$h_\varepsilon(x) = \left(\frac{(\cdot)^2}{2\varepsilon} \square |\cdot| \right)(x)$$

Proof. Exercise. □

Using the above result the convex conjugate h_ε^* is given as:

$$h_\varepsilon^*(y) = \frac{\varepsilon}{2} y^2 + \iota_{|\cdot| \leq 1}(y)$$