Chapter 5

Operator Splitting Methods

Convex Optimization for Computer Vision
SS 2016

Michael Moeller
Thomas Möllenhoff
Emanuel Laude
Computer Vision Group
Department of Computer Science
TU München

Relations

Monotone Operators

Fixed Point Iterations

Proximal Point Algorithm

PDHG Revisited

Douglas-Rachford Splitting

Recap and Motivation

- **Last 3 lectures**: PDHG method for minimizing structured convex problems

$$\min_{u \in \mathbb{R}^n} \; G(u) + F(Ku)$$

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- **Last 3 lectures**: PDHG method for minimizing structured convex problems

$$\min_{u \in \mathbb{R}^n} \; G(u) + F(Ku)$$

- Unintuitive overrelaxation, rather involved convergence analysis

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- Unintuitive overrelaxation, rather involved convergence analysis
- Next lectures: simple and unified convergence analysis of many different algorithms within a single approach

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- Unintuitive overrelaxation, rather involved convergence analysis
- Next lectures: simple and unified convergence analysis of many different algorithms within a single approach
- Key ideas: monotone operators, fixed point iterations

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- Unintuitive overrelaxation, rather involved convergence analysis
- Next lectures: simple and unified convergence analysis of many different algorithms within a single approach
- Key ideas: monotone operators, fixed point iterations
- Give a new understanding of convex optimization algorithms

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Notation

- A relation R on $\mathbb{R}^n$ is a subset of $\mathbb{R}^n \times \mathbb{R}^n$

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- A relation R on $\mathbb{R}^n$ is a subset of $\mathbb{R}^n \times \mathbb{R}^n$
- We will refer to it as a set-valued **operator** and overload the usual matrix notation

$$R(x) = Rx := \{y \in \mathbb{R}^n \mid (x, y) \in R\}.$$

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- If Rx is a singleton or empty for all x, then R is a function (or single-valued operator) with domain

$$\text{dom}(R) := \{x \in \mathbb{R}^n \mid Rx \neq \emptyset\}$$

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- Abuse of notation: identify singleton $\{x\}$ with x, i.e., write $Rx = y$ instead of $Rx \ni y$ if R is function

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- Concept: identifying functions with their *graph*

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Some Examples

- Empty relation: $\emptyset$

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Some Examples

- Empty relation: $\emptyset$
- Identity: $I := \{(u, u) \mid u \in \mathbb{R}^n\}$

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Some Examples

- Empty relation: $\emptyset$
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- Gradient relation:

$$\nabla E := \{(u, \nabla E(u)) \mid u \in \mathbb{R}^n\}$$

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- Subdifferential relation:

$$\partial E := \{(u,g) \mid u \in \text{dom}(E), E(v) \geq E(u) + \langle g, v-u \rangle, \forall v \in \mathbb{R}^n\}$$

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- Another possible view: think of relations as a set valued functions, e.g., $\partial E : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$

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Our Goal

Solve generalized equation (inclusion) problem

$$0 \in R(u)$$

i.e., find $u \in \mathbb{R}^n$ such that $(u, 0) \in R$.

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- Set $R = \partial E$, then the goal is to find $0 \in \partial E(u)$

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- This are just the optimality conditions of our prototypical optimization problem:

$$\arg\min_{u \in \mathbb{R}^n} \; E(u)$$

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- Finding saddle-points $(\tilde{u}, \tilde{p})$ of

$$PD(u, p) = G(u) - F^*(p) + \langle Ku, p \rangle$$

corresponds to the inclusion problem

$$0 \in \begin{bmatrix} \partial G & K^T \\ -K & \partial F^* \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix}$$

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Operations on Relations

- Inverse $R^{-1} = \{(y, x) \mid (x, y) \in R\}$
  - Exists for *any* relation
  - Reduces to inverse function when R is injective function

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- Addition $R + S = \{(x, y + z) \mid (x, y) \in R, (x, z) \in S\}$

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- $J_R = \{(x + \lambda y, x) \mid (x, y) \in R\}$

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$\rightarrow$ **Draw a picture for $E(u) = |u|$**

Monotone Operators

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Monotone Operators

Definition

The set-valued operator $T \subset \mathbb{R}^n \times \mathbb{R}^n$ is called **monotone** if

$$\langle u - v, Tu - Tv \rangle \geq 0, \ \forall u, v \in \mathbb{R}^n. \qquad \text{Notation}^1$$

An operator T is called **maximally monotone** if it is not contained in any other monotone operator.

[1] This is again abuse of notation for $\langle u - v, p - q \rangle \geq 0, \ \forall p \in Tu, \forall q \in Tv$

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Examples of monotone operators:

- Monotonically non-decreasing functions $T : \mathbb{R} \to \mathbb{R}$

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- Monotonically non-decreasing functions $T : \mathbb{R} \to \mathbb{R}$
- Any positive semi-definite matrix A: $\langle Ax - Ay, x - y \rangle \geq 0$

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- Proximity operators of convex functions $\text{prox}_{\tau f} : \mathbb{R}^n \to \mathbb{R}^n$

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Monotone Operators

Calculus rules (exercise):

- T monotone, $\lambda \geq 0 \Rightarrow \lambda T$ monotone

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- T monotone $\Rightarrow T^{-1}$ monotone

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Some important definitions/properties:

- Lipschitz operators (and in particular nonexpansive operators) are single-valued (functions)

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Some important definitions/properties:

- Lipschitz operators (and in particular nonexpansive operators) are single-valued (functions)
- x is called *fixed point* of operator T if $x = Tx$
- If F is nonexpansive (Lipschitz constant $L \leq 1$) and $\operatorname{dom} T = \mathbb{R}^n$ then the set of fixed points $(I - F)^{-1}(0)$ is closed and convex **(exercise)**

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Resolvent and Cayley Operators

- Let $T \subset \mathbb{R}^n \times \mathbb{R}^n$ be set-valued operator

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Resolvent and Cayley Operators

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- The *resolvent operator* of T is given as $J_{\lambda T} := (I + \lambda T)^{-1}$

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- Special case: $T = \partial f$, $J_{\lambda \partial f}$ is proximal operator of f

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- From previous slide: resolvent is monotone if T is monotone

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- The *Cayley operator* (or reflection operator) of T is defined as $C_{\lambda T} := 2J_{\lambda T} - I$

Resolvent and Cayley Operators

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- Special case: $T = \partial f$, $J_{\lambda \partial f}$ is proximal operator of f
- From previous slide: resolvent is monotone if T is monotone
- The *Cayley operator* (or reflection operator) of T is defined as $C_{\lambda T} := 2J_{\lambda T} - I$

Facts:

- $0 \in Tx$ if and only if $x = J_{\lambda T}x = C_{\lambda T}x$

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Resolvent and Cayley Operators

- Let $T \subset \mathbb{R}^n \times \mathbb{R}^n$ be set-valued operator
- The *resolvent operator* of T is given as $J_{\lambda T} := (I + \lambda T)^{-1}$
- Special case: $T = \partial f$, $J_{\lambda \partial f}$ is proximal operator of f
- From previous slide: resolvent is monotone if T is monotone
- The *Cayley operator* (or reflection operator) of T is defined as $C_{\lambda T} := 2J_{\lambda T} - I$

Facts:

- $0 \in Tx$ if and only if $x = J_{\lambda T}x = C_{\lambda T}x$
- If T is monotone, then $J_{\lambda T}$ and $C_{\lambda T}$ are nonexpansive

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The Main Algorithm

- Recall that $u \in \mathbb{R}^n$ is fixed point of $F : \mathbb{R}^n \to \mathbb{R}^n$, if $u = Fu$

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The Main Algorithm

- Recall that $u \in \mathbb{R}^n$ is fixed point of $F : \mathbb{R}^n \to \mathbb{R}^n$, if $u = Fu$
- The main algorithm of this chapter is the *fixed point* or *Picard iteration* for some given $u^0 \in \mathbb{R}^n$:

$$u^{k+1} = Fu^k, \qquad k = 0, 1, 2, \dots$$

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- We will see that many important convex optimization algorithms can be written in this form

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- Allows simple and unified analysis

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Iteration of Contraction Mappings

Contraction Mapping Theorem

Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is a contraction with Lipschitz constant $L < 1$. Then the fixed point iteration

$$u^{k+1} = Fu^k,$$

also called contraction mapping algorithm, converges to the unique fixed point of F.

$\rightarrow$ Proof: see literature[2]

[2]This theorem is also known as the Banach fixed point theorem.

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- Example: the gradient method can be written as

$$u^{k+1} = (I - \tau \nabla E)u^k$$

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- Suppose E is m-strongly convex and L-smooth, then $I - \tau \nabla E$ is Lipschitz with $L_{GM} = \max\{|1 - \tau m|, |1 - \tau L|\}$

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- Suppose E is m-strongly convex and L-smooth, then $I - \tau\nabla E$ is Lipschitz with $L_{GM} = \max\{|1 - \tau m|, |1 - \tau L|\}$
- $I - \tau\nabla E$ is contractive for $\tau \in (0, 2/L)$

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Iteration of Averaged Nonexpansive Mappings

- Recall that a mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is called *nonexpansive* if it is Lipschitz with constant $L \leq 1$.

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Iteration of Averaged Nonexpansive Mappings

- Recall that a mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is called *nonexpansive* if it is Lipschitz with constant $L \leq 1$.
- Fixed point iteration of nonexpansive mapping doesn't necessarily converge (example: rotation, reflection)

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Iteration of Averaged Nonexpansive Mappings

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- The mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is called *averaged* if $F = (1 - \theta)I + \theta T$, for some nonexpansive operator T and $\theta \in (0, 1)$

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Theorem: Krasnosel'skii-Mann

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be averaged, and denote the (non-empty) set of fixed points of F as U. Then the sequence (u^k) produced by the iteration

$$u^{k+1} = Fu^k$$

converges to a fixed point $u^* \in U$, i.e., $u^k \to u^*$.

$\to$ Proof: board!

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Example: gradient method

- Assume E is L-smooth but not strongly convex

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- Possible to show that the operator $(I - \tau \nabla E)$ is Lipschitz continuous with parameter $L_{GM} = \max\{1, |1 - \tau L|\}$

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- For $0 < \tau \leq 2/L$, this operator is nonexpansive
- It is also averaged for $0 < \tau < 2/L$ since

$$(I - \tau \nabla E) = (1 - \theta)I + \theta(I - (2/L)\nabla E),$$

with $\theta = \tau L/2 < 1$.

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  with $\theta = \tau L/2 < 1$.
- Hence, we get convergence of the gradient descent method from the previous theorem

Proximal Point Algorithm

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The Proximal Point Algorithm

- Recall our original goal of finding $u \in \mathbb{R}^n$ with

$$0 \in Tu,$$

for $T \subset \mathbb{R}^n \times \mathbb{R}^n$ monotone.

[3] R. T. Rockafellar, Monotone Operators and the Proximal Point Algorithm, SIAM J. Control and Optimization, 1976

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Definition: Proximal Point Algorithm (PPA) [3]

Given some maximally monotone operator $T \subset \mathbb{R}^n \times \mathbb{R}^n$, and some sequence $(\lambda_k) > 0$. Then the iteration

$$u^{k+1} = (I + \lambda_k T)^{-1} u^k,$$

is called the *proximal point algorithm*.

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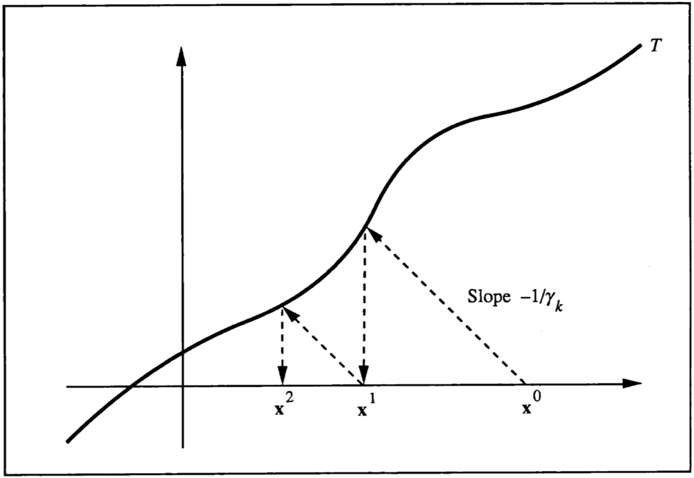
Intuition of the Proximal Point Algorithm [4]



[4] Eckstein, Splitting methods for monotone operators with applications to parallel optimzation, 1989, pp. 42

Convergence of Proximal Point Algorithm

- The resolvent $J_{\lambda T} = (I + \lambda T)^{-1}$ is an averaged operator

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Convergence of Proximal Point Algorithm

- The resolvent $J_{\lambda T} = (I + \lambda T)^{-1}$ is an averaged operator
- To see this, consider the *reflection* or *Cayley* operator

$$C_{\lambda T} := 2J_{\lambda T} - I \Leftrightarrow J_{\lambda T} = \frac{1}{2} I + \frac{1}{2} C_{\lambda T}$$

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- Hence $J_{\lambda T}$ is averaged with $\theta = \frac{1}{2}$, as we have seen in the last lecture that $C_{\lambda T}$ is nonexpansive
- Proximal Point algorithm converges as it is fixed point iteration of averaged operator

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PDHG as Proximal Point Method

- Remember that for convex-concave saddle point problems

$$PD(u,p) = G(u) - F^*(p) + \langle Ku, p\rangle$$

we have the following:

$$(\tilde{u},\tilde{p}) = \arg\operatorname{minmax}_{u,p} PD(u,p) \Leftrightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \underbrace{\begin{bmatrix} \partial G(\tilde{u}) + K^T\tilde{p} \\ -K\tilde{u} + \partial F^*(\tilde{p}) \end{bmatrix}}_{=:T(\tilde{u},\tilde{p})}$$

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- For convex F^* and G, T is monotone
- Idea: use the proximal point to find zero of T
- Stack primal and dual variables into vector $z = (u,p)^T$:

$$z^{k+1} = (I + \lambda T)^{-1} z^k \;\Leftrightarrow\; z^k - z^{k+1} \in \lambda T z^{k+1}$$

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- Plugging things in yields

$$u^k - u^{k+1} \in \lambda \partial G(u^{k+1}) + \lambda K^T p^{k+1}$$
$$p^k - p^{k+1} \in \lambda \partial F^*(p^{k+1}) - \lambda K u^{k+1}$$

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PDHG as Proximal Point Method

- Reformulating the following

$$0 \in \lambda^{-1} \begin{bmatrix} u^{k+1} - u^k \\ p^{k+1} - p^k \end{bmatrix} + \underbrace{\begin{bmatrix} \partial G(u^{k+1}) + K^T p^{k+1} \\ \partial F^*(p^{k+1}) - K u^{k+1} \end{bmatrix}}_{=: T(\tilde{u}, \tilde{p})}$$

leads to:

$$\begin{aligned} u^{k+1} &= (I + \lambda \partial G)^{-1}(u^k - \lambda K^T p^{k+1}) \\ &= \text{prox}_{\lambda G}(u^k - \lambda K^T p^{k+1}) \\ p^{k+1} &= (I + \lambda \partial F^*)^{-1}(p^k + \lambda K u^{k+1}) \\ &= \text{prox}_{\lambda F^*}(p^k + \lambda K u^{k+1}) \end{aligned}$$

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- Almost looks like the PDHG method, step size λ

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- Almost looks like the PDHG method, step size λ
- **Problem:** cannot implement this algorithm, since updates in u^{k+1} and p^{k+1} depend on each other

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PDHG as Proximal Point Method

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PDHG as Proximal Point Method

- Consider the following:

$$0 \in M \begin{bmatrix} u^{k+1} - u^k \\ p^{k+1} - p^k \end{bmatrix} + \underbrace{\begin{bmatrix} \partial G(u^{k+1}) + K^T p^{k+1} \\ \partial F^*(p^{k+1}) - K u^{k+1} \end{bmatrix}}_{=: T(\tilde{u}, \tilde{p})}$$

- Step size $M \in \mathbb{R}^{(n+m)\times(n+m)}$ is now a matrix

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- Step size $M \in \mathbb{R}^{(n+m)\times(n+m)}$ is now a matrix
- Take the following choice

$$M = \begin{bmatrix} \frac{1}{\tau} I & -K^T \\ -\theta K & \frac{1}{\sigma} I \end{bmatrix}$$

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PDHG as Proximal Point Method

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- Allows to recover PDHG as proximal point algorithm (PPA)

$$\begin{aligned} u^{k+1} &= \text{prox}_{\tau G}(u^k - \tau K^T p^k), \\ p^{k+1} &= \text{prox}_{\sigma F^*}(p^k + \sigma K(u^{k+1} + \theta(u^{k+1} - u^k))) \end{aligned}$$

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- Step size $M \in \mathbb{R}^{(n+m)\times(n+m)}$ is now a matrix
- Take the following choice

$$M = \begin{bmatrix} \frac{1}{\tau} I & -K^T \\ -\theta K & \frac{1}{\sigma} I \end{bmatrix}$$

- Allows to recover PDHG as proximal point algorithm (PPA)

$$\begin{aligned} u^{k+1} &= \text{prox}_{\tau G}(u^k - \tau K^T p^k), \\ p^{k+1} &= \text{prox}_{\sigma F^*}(p^k + \sigma K(u^{k+1} + \theta(u^{k+1} - u^k))) \end{aligned}$$

- This is called generalized or customized PPA:

$$0 \in M(z^{k+1} - z^k) + T z^{k+1} \;\Leftrightarrow\; z^{k+1} = (M + T)^{-1} M z^k$$

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Convergence of Customized Proximal Point Method

- For symmetric, positive definite M, we can write $M = L^T L$, L invertible (Cholesky decomposition)

[5] Bauschke, Combettes, Convex Analysis and Monotone Operator Theory in Hilbert Spaces, Theorem 24.5

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Convergence of Customized Proximal Point Method

- For symmetric, positive definite M, we can write $M = L^T L$, L invertible (Cholesky decomposition)
- Apply classical PPA to operator $T' = L^{-T} \circ T \circ L^{-1}$

$$y^{k+1} = (I + L^{-T} \circ T \circ L^{-1})^{-1} y^k$$

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- T (maximally) monotone $\Rightarrow L^{-T} \circ T \circ L^{-1}$ (maximally) monotone [5]

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- Define $Lx = y$, then $0 \in (L^{-T} \circ T \circ L^{-1})y \Leftrightarrow 0 \in Tx$

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- Writing out the algorithm in terms of x yields

$$0 \in M(x^{k+1} - x^k) + Tx^{k+1}$$

[5] Bauschke, Combettes, Convex Analysis and Monotone Operator Theory in Hilbert Spaces, Theorem 24.5

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- Writing out the algorithm in terms of x yields

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- Hence customized PPA inherits convergence from classical proximal point

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Convergence of PDHG

- When is the step size matrix symmetric positive definite?

$$M = \begin{bmatrix} \frac{1}{\tau} I & -K^T \\ -\theta K & \frac{1}{\sigma} I \end{bmatrix}$$

[6] T. Pock, A. Chambolle, Diagonal Preconditioning for first-order primal-dual algorithms in convex optimization, ICCV 2011

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$$M = \begin{bmatrix} \frac{1}{\tau} I & -K^T \\ -\theta K & \frac{1}{\sigma} I \end{bmatrix}$$

- Step size requirement for PDHG is $\tau\sigma \|K\|^2 < 1$, $\tau\sigma > 0$

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Lemma (Pock-Chambolle-2011 [6])

Let $\theta = 1$, T and Σ symmetric positive definite maps satisfying

$$\left\| \Sigma^{\frac{1}{2}} K \mathrm{T}^{\frac{1}{2}} \right\|^2 < 1,$$

then the block matrix

$$M = \begin{bmatrix} \mathrm{T}^{-1} & -K^T \\ -\theta K & \Sigma^{-1} \end{bmatrix}$$

is symmetric and positive definite.

[6] T. Pock, A. Chambolle, Diagonal Preconditioning for first-order primal-dual algorithms in convex optimization, ICCV 2011

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Summary

- Customized proximal point algorithms yield a whole family of methods, many choices of M are concievable

$$0 \in M(z^{k+1} - z^k) + Tz^{k+1}$$

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- PDHG corresponds to one particular choice of M

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- Overrelaxation with $\theta = 1$ required to make M symmetric

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- Classical proximal point converges as it is fixed point iteration of averaged operator

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- Classical proximal point converges as it is fixed point iteration of averaged operator
- **Next lecture:** Douglas-Rachford splitting and ADMM

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Organizational Remarks

Exams:

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Remaining lectures:

- Next Monday 20.06. hints for getting started with the optimization challenge!

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- 22.06. Some practical considerations of PDHG/ADMM

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- 27.06. - 01.07. no lecture / exercises, repeat and review what you have learned!

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- Last lecture on 13.07. repeat of content, questions

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Douglas-Rachford Splitting

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Motivation

- Last lecture: proximal point algorithm for finding the zero of a monotone operator T

$$0 \in Tu \;\Leftrightarrow\; u = (I + \lambda T)^{-1} u$$

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Motivation

- Last lecture: proximal point algorithm for finding the zero of a monotone operator T

$$0 \in Tu \;\Leftrightarrow\; u = (I + \lambda T)^{-1} u$$

- Often the resolvent $J_{\lambda T} := (I + \lambda T)^{-1}$ is hard to compute

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- Often the resolvent $J_{\lambda T} := (I + \lambda T)^{-1}$ is hard to compute
- One remedy: matrix-valued step-size / customized PPA

$$u^{k+1} = (M + T)^{-1} M u^k$$

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- They exploit further structure of the problem:

$$T = A + B$$

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- Another possibility are **splitting methods**
- They exploit further structure of the problem:

$$T = A + B$$

- Resolvents $J_{\lambda A} = (I + \lambda A)^{-1}$ and $J_{\lambda B} = (I + \lambda B)^{-1}$ can be more easily evaluated than $J_{\lambda T}$

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Splitting methods

- $T = A + B$, A and B maximal monotone

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Splitting methods

- $T = A + B$, A and B maximal monotone
- Cayley operators $C_A = 2J_A - I$ and $C_B = 2J_A - I$ are nonexpansive

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- **Main result:** ($\rightarrow$ board!)

$$0 \in Au + Bu \;\Leftrightarrow\; C_A C_B v = v,\; u = J_B v$$

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$$0 \in Au + Bu \;\Leftrightarrow\; C_A C_B v = v,\; u = J_B v$$

- Hence, solutions can be found from fixed point of the operator $C_A C_B$

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$\rightarrow$ **Draw a picture for** $T = \partial \iota_{C_1} + \partial \iota_{C_2}$!

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Douglas-Rachford Splitting

Splitting Methods

- *Peaceman-Rachford* splitting is undamped iteration

$$v^{k+1} = C_A C_B v^k$$

[7] J. Douglas, H. H. Rachford, On the numerical solution of heat conduction problems in two and three space variables. Transactions of the AMS, 1956.

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Splitting Methods

- *Peaceman-Rachford* splitting is undamped iteration

$$v^{k+1} = C_A C_B v^k$$

- Doesn't converge in the general case, needs either C_A or C_B to be a contraction

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- *Douglas-Rachford* splitting [7] is the damped iteration

$$v^{k+1} = \left(\frac{1}{2} I + \frac{1}{2} C_A C_B\right) v^k,$$

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- Recover solution by $u^* = J_B v^*$

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$$v^{k+1} = \left(\frac{1}{2} I + \frac{1}{2} C_A C_B \right) v^k,$$

- Recover solution by $u^* = J_B v^*$
- Always converges if there exists a solution $0 \in Au^* + Bu^*$, since it's fixed point iteration of averaged operator

---

[7] J. Douglas, H. H. Rachford, On the numerical solution of heat conduction problems in two and three space variables. Transactions of the AMS, 1956.

Douglas-Rachford Splitting (DRS)

- The Douglas-Rachford iteration $v^{k+1} = \left(\frac{1}{2} I + \frac{1}{2} C_A C_B\right) v^k$ can be written as

$$
\begin{aligned}
u_b^{k+1} &= J_B(v^k), \\
\tilde{v}^{k+1} &= 2u_b^{k+1} - v^k, \\
u_a^{k+1} &= J_A(\tilde{v}^{k+1}), \\
v^{k+1} &= v^k + u_a^{k+1} - u_b^{k+1}.
\end{aligned}
$$

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- u_a^k and u_b^k can be thought of estimates to a solution

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v^{k+1} &= v^k + u_a^{k+1} - u_b^{k+1}.
\end{aligned}
$$

- u_a^k and u_b^k can be thought of estimates to a solution
- v^k running sum of residuals, drives u_a^k and u_b^k together

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Application to Convex Optimization

- Let's apply DRS to minimize

$$\min_{u \in \mathbb{R}^n} \; G(u) + F(u)$$

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Application to Convex Optimization

- Let's apply DRS to minimize

$$\min_{u \in \mathbb{R}^n} G(u) + F(u)$$

- $G : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$, $F : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ closed, proper, cvx.

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Application to Convex Optimization

- Let's apply DRS to minimize

$$\min_{u \in \mathbb{R}^n} \; G(u) + F(u)$$

- $G : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$, $F : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ closed, proper, cvx.
- Optimality conditions (assuming $\text{ri}(\text{dom}G) \cap \text{ri}(\text{dom}F) \neq \emptyset$):

$$0 \in \tau \partial G(u) + \tau \partial F(u)$$

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$$0 \in \tau\partial G(u) + \tau\partial F(u)$$

- Find zero of $T = A + B$, $A = \tau\partial G$, $B = \tau\partial F$
- The algorithm becomes (after slight simplifications):

$$\begin{aligned} u^{k+1} &= \text{prox}_{\tau G}(v^k), \\ v^{k+1} &= \text{prox}_{\tau F}(2u^{k+1} - v^k) + v^k - u^{k+1}. \end{aligned}$$

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Reformulation of DRS

- We can rewrite the step in v^{k+1} using Moreau's Identity

$$\begin{aligned} u^{k+1} &= \text{prox}_{\tau G}(v^k), \\ v^{k+1} &= \text{prox}_{\tau F}(2u^{k+1} - v^k) + v^k - u^{k+1} \\ &= u^{k+1} + \tau \text{prox}_{(1/\tau)F^*}((2u^{k+1} - v^k)/\tau) \end{aligned}$$

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- Introduce variable $p^k = \frac{u^k - v^k}{\tau} \Leftrightarrow v^k = u^k - \tau p^k$, $\sigma = 1/\tau$

$$\begin{aligned} u^{k+1} &= \text{prox}_{\tau G}(u^k - \tau p^k), \\ p^{k+1} &= \text{prox}_{\sigma F^*}(p^k + \sigma(2u^{k+1} - u^k)) \end{aligned}$$

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- Looks familiar? :-)

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\end{aligned}
$$

- Looks familiar? :-)
- Applying DRS on the primal problem $\min_u G(u) + F(u)$ is equivalent to PDHG!

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Optimization Problems with Compositions

- Ideally we'd like to solve problems of the form

$$\min_u \; G(u) + F(w), \quad \text{s.t.} \quad w = Ku$$

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Optimization Problems with Compositions

- Ideally we'd like to solve problems of the form

$$\min_u \; G(u) + F(w), \quad \text{s.t.} \quad w = Ku$$

- In many applications we would actually like to minimize

$$\min_u \; G(u) + \sum_{i=1}^{N} F_i(K_i u)$$

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- Rewrite using trick:

$$w = \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix}, K = \begin{bmatrix} K_1 \\ \dots \\ K_N \end{bmatrix}, \quad \to F(w) = \sum_{i=1}^{N} F_i(w_i)$$

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- Virtually any convex optimization problem fits into this form

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- Virtually any convex optimization problem fits into this form
- Even problems looking very complicated at first glance can be split up into many simple substeps

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Thomas Möllenhoff
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Option 1: Graph Projection Splitting

- We want to minimize for $K : \mathbb{R}^n \to \mathbb{R}^m$

$$\min_{u \in \mathbb{R}^n, w \in \mathbb{R}^m} \; G(u) + F(w) \quad \text{s.t.} \quad Ku = w$$

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- Rewrite problem using $(u, w) \in \mathbb{R}^{n+m}$ as

$$\min_{u,w} \tilde{G}(u, w) + \tilde{F}(u, w)$$

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- Rewrite problem using $(u, w) \in \mathbb{R}^{n+m}$ as

$$\min_{u,w} \tilde{G}(u, w) + \tilde{F}(u, w)$$

- Set $\tilde{G}(u, w) = G(u) + F(w)$

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$$\min_{u,w} \tilde{G}(u, w) + \tilde{F}(u, w)$$

- Set $\tilde{G}(u, w) = G(u) + F(w)$
- Set $\tilde{F}(u, w) = \begin{cases} 0, & \text{if } Ku = w \\ \infty, & \text{else.} \end{cases}$

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- Proximal operator for $\tilde{G}$ is simple if proximal operators for F and G are simple

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- Rewrite problem using $(u, w) \in \mathbb{R}^{n+m}$ as

$$\min_{u,w} \tilde{G}(u, w) + \tilde{F}(u, w)$$

- Set $\tilde{G}(u, w) = G(u) + F(w)$
- Set $\tilde{F}(u, w) = \begin{cases} 0, & \text{if } Ku = w \\ \infty, & \text{else.} \end{cases}$
- Proximal operator for $\tilde{G}$ is simple if proximal operators for F and G are simple
- Proximal operator for $\tilde{F}$ is projection onto the graph of $Ku = w$ (solving a least squares problem)

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Option 1: Graph Projection Splitting

- Iterations can be written as [8]

$$(u^{k+1/2}, w^{k+1/2}) = \left(\text{prox}_G(u^k - \tilde{u}^k), \text{prox}_F(w^k - \tilde{w}^k)\right),$$

$$(u^{k+1}, w^{k+1}) = \Pi(u^{k+1/2} + \tilde{u}^k, w^{k+1/2} + \tilde{w}^k),$$

$$(\tilde{u}^{k+1}, \tilde{w}^{k+1}) = (\tilde{u}^k + u^{k+1/2} - u^{k+1}, \tilde{w}^k + w^{k+1/2} - w^{k+1}).$$

[8] N. Parikh, S. Boyd, Block Splitting for Distributed Optimization, 2014

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\end{aligned}
$$

- Projection is given as:

$$
\Pi(c, d) = A^{-1} \begin{bmatrix} c + A^T d \\ 0 \end{bmatrix}, A = \begin{bmatrix} I & K^T \\ K & -I \end{bmatrix}
$$

[8] N. Parikh, S. Boyd, Block Splitting for Distributed Optimization, 2014

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- Can use (preconditioned) conjugate gradient to approximately compute projection

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- Important: warm-start linear system solver with solution from previous iteration

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$$

- Can use (preconditioned) conjugate gradient to approximately compute projection
- Important: warm-start linear system solver with solution from previous iteration
- Other possibility: factorization caching

[8] N. Parikh, S. Boyd, Block Splitting for Distributed Optimization, 2014

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Option 2: DRS for Problems with Compositions

- Consider the dual problem to $\min_u \; G(u) + F(Ku)$

$$\min_p \; G^*(-K^*p) + F^*(p) = (G^* \circ -K^*)(p) + F^*(p)$$

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$$u^{k+1} = \text{prox}_{\sigma(G^* \circ -K^*)}(v^k),$$
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- Reorder slightly with new variable w^{k+1}

$$\begin{aligned} u^{k+1} &= \text{prox}_{\sigma(G^* \circ -K^*)}(v^k), \\ p^{k+1} &= \text{prox}_{\sigma F^*}(2u^{k+1} - v^k), \\ v^{k+1} &= p^{k+1} + v^k - u^{k+1} \end{aligned}$$

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Option 2: DRS for Problems with Compositions

- The prox involving the composition is given by:

$$\text{prox}_{\sigma(G^* \circ -K^*)}(v) = v + \sigma K \operatorname*{argmin}_u G(u) + \frac{\sigma}{2} \left\| Ku + \frac{v}{\sigma} \right\|^2$$

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- Often expensive or difficult to evaluate due to the Ku-term

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- Often expensive or difficult to evaluate due to the Ku-term
- Iteration can be written as

$$\begin{aligned} u^{k+1} &= \operatorname*{argmin}_u G(u) + \frac{\sigma}{2} \left\| Ku + \frac{v^k}{\sigma} \right\|^2, \\ \tilde{u}^{k+1} &= v^k + \sigma K u^{k+1}, \\ p^{k+1} &= \text{prox}_{\sigma F^*}(2\tilde{u}^{k+1} - v^k), \\ v^{k+1} &= p^{k+1} + v^k - \tilde{u}^{k+1} \end{aligned}$$

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Option 2: DRS for Problems with Compositions

- The prox involving the composition is given by:

$$\text{prox}_{\sigma(G^* \circ -K^*)}(v) = v + \sigma K \operatorname*{argmin}_u G(u) + \frac{\sigma}{2} \left\| Ku + \frac{v}{\sigma} \right\|^2$$

- Often expensive or difficult to evaluate due to the *Ku*-term
- Iteration can be written as

$$\begin{aligned} u^{k+1} &= \operatorname*{argmin}_u G(u) + \frac{\sigma}{2} \left\| Ku + \frac{v^k}{\sigma} \right\|^2, \\ \tilde{u}^{k+1} &= v^k + \sigma K u^{k+1}, \\ p^{k+1} &= \text{prox}_{\sigma F^*}(2\tilde{u}^{k+1} - v^k), \\ v^{k+1} &= p^{k+1} + v^k - \tilde{u}^{k+1} \end{aligned}$$

- Alternatively this can be simplified to

$$\begin{aligned} u^{k+1} &= \operatorname*{argmin}_u G(u) + \frac{\sigma}{2} \left\| Ku + \frac{v^k}{\sigma} \right\|^2, \\ p^{k+1} &= \text{prox}_{\sigma F^*}(v^k + 2\sigma K u^{k+1}), \\ v^{k+1} &= p^{k+1} - \sigma K u^{k+1} \end{aligned}$$

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Option 2: DRS for Problems with Compositions

- Even more simple:

$$
\begin{aligned}
u^{k+1} &= \underset{u}{\operatorname{argmin}}\, G(u) + \frac{\sigma}{2}\left\| Ku + \frac{p^k - \sigma K u^k}{\sigma} \right\|^2, \\
p^{k+1} &= \operatorname{prox}_{\sigma F^*}(p^k + \sigma K(2u^{k+1} - u^k)),
\end{aligned}
$$

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Option 2: DRS for Problems with Compositions

- Even more simple:

$$u^{k+1} = \operatorname*{argmin}_u G(u) + \frac{\sigma}{2}\left\| Ku + \frac{p^k - \sigma K u^k}{\sigma}\right\|^2,$$
$$p^{k+1} = \text{prox}_{\sigma F^*}(p^k + \sigma K(2u^{k+1} - u^k)),$$

- Optimality conditions for the iterates:

$$0 \in \partial G(u^{k+1}) + \sigma K^T(Ku^{k+1} + \frac{1}{\sigma}(p^k - \sigma K u^k))$$
$$0 \in \partial F^*(p^{k+1}) + \frac{1}{\sigma}(p^{k+1} - p^k - \sigma K 2u^{k+1} + \sigma K u^k)$$

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Option 2: DRS for Problems with Compositions

- Even more simple:

$$u^{k+1} = \operatorname*{argmin}_u G(u) + \frac{\sigma}{2}\left\|Ku + \frac{p^k - \sigma K u^k}{\sigma}\right\|^2,$$
$$p^{k+1} = \operatorname{prox}_{\sigma F^*}(p^k + \sigma K(2u^{k+1} - u^k)),$$

- Optimality conditions for the iterates:

$$0 \in \partial G(u^{k+1}) + \sigma K^T(Ku^{k+1} + \frac{1}{\sigma}(p^k - \sigma K u^k))$$
$$0 \in \partial F^*(p^{k+1}) + \frac{1}{\sigma}(p^{k+1} - p^k - \sigma K 2u^{k+1} + \sigma K u^k)$$

- Adding and substracting $K^T p^{k+1}$ to first line yields

$$0 \in \partial G(u^{k+1}) + K^T p^{k+1} + \sigma K^T K(u^{k+1} - u^k) - K^T(p^{k+1} - p^k)$$
$$0 \in \partial F^*(p^{k+1}) - Ku^{k+1} - K(u^{k+1} - u^k) + \frac{1}{\sigma}(p^{k+1} - p^k)$$

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Relation to PDHG

- Previous iterations can be written as PPA, $z = (u, p)^T$:

$$0 \in \underbrace{\begin{bmatrix} \partial G & K^T \\ -K & \partial F^* \end{bmatrix} \begin{bmatrix} u^{k+1} \\ p^{k+1} \end{bmatrix}}_{Tz^{k+1}} + \underbrace{\begin{bmatrix} \sigma K^T K & -K^T \\ -K & \frac{1}{\sigma} I \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} u^{k+1} - u^k \\ p^{k+1} - p^k \end{bmatrix}}_{z^{k+1} - z^k}$$

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- Matrix M only positive semidefinite, our convergence result for Proximal Point algorithm does not apply directly

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- Matrix M only positive semidefinite, our convergence result for Proximal Point algorithm does not apply directly
- PDHG with $\theta = 1$ can be seen as inexact/approximative DRS,

$$\sigma K^T K \approx \frac{1}{\tau} I$$

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- Often makes iterations much cheaper
- For semi-orthogonal ($K^T K = \nu I$) this approximation is exact

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Alternating Direction Method of Multipliers (ADMM)

- Recall this formulation

$$u^{k+1} = \operatorname*{argmin}_u G(u) + \frac{\sigma}{2}\left\| Ku + \frac{v^k}{\sigma} \right\|^2,$$
$$p^{k+1} = \text{prox}_{\sigma F^*}(v^k + 2\sigma K u^{k+1}),$$
$$v^{k+1} = p^{k+1} - \sigma K u^{k+1}$$

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$$p^{k+1} = \text{prox}_{\sigma F^*}(v^k + 2\sigma K u^{k+1}),$$

$$v^{k+1} = p^{k+1} - \sigma K u^{k+1}$$

- Apply Moreau's identity to step in p^{k+1}

$$u^{k+1} = \operatorname*{argmin}_u G(u) + \frac{\sigma}{2}\left\| Ku + \frac{v^k}{\sigma} \right\|^2,$$

$$p^{k+1} = v^k + 2\sigma K u^{k+1} - \sigma \text{prox}_{\sigma F}(\frac{v^k}{\sigma} + 2Ku^{k+1}),$$

$$v^{k+1} = p^{k+1} - \sigma K u^{k+1}$$

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Alternating Direction Method of Multipliers (ADMM)

- Make new variable for $\text{prox}_{\sigma F}$-step, write prox as argmin:

$$
\begin{aligned}
u^{k+1} &= \underset{u}{\text{argmin}}\, G(u) + \frac{\sigma}{2}\left\| Ku + \frac{v^k}{\sigma} \right\|^2, \\
w^{k+1} &= \underset{w}{\text{argmin}}\, F(w) + \frac{\sigma}{2}\left\| w - \frac{v^k}{\sigma} - 2Ku^{k+1} \right\|^2, \\
p^{k+1} &= v^k + 2\sigma K u^{k+1} - \sigma w^{k+1}, \\
v^{k+1} &= p^{k+1} - \sigma K u^{k+1}
\end{aligned}
$$

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Alternating Direction Method of Multipliers (ADMM)

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$$
\begin{aligned}
u^{k+1} &= \underset{u}{\text{argmin}}\, G(u) + \frac{\sigma}{2}\left\|Ku + \frac{v^k}{\sigma}\right\|^2, \\
w^{k+1} &= \underset{w}{\text{argmin}}\, F(w) + \frac{\sigma}{2}\left\|w - \frac{v^k}{\sigma} - 2Ku^{k+1}\right\|^2, \\
p^{k+1} &= v^k + 2\sigma Ku^{k+1} - \sigma w^{k+1}, \\
v^{k+1} &= p^{k+1} - \sigma Ku^{k+1}
\end{aligned}
$$

- Replacing the variable v^k in the u^{k+1} update yields

$$
u^{k+1} = \underset{u}{\text{argmin}}\, G(u) + \frac{\sigma}{2}\left\|Ku + \frac{p^k - \sigma Ku^k}{\sigma}\right\|^2,
$$

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Alternating Direction Method of Multipliers (ADMM)

- Replace variable p^k in all update steps

$$u^{k+1} = \operatorname*{argmin}_u G(u) + \frac{\sigma}{2}\left\| Ku + \frac{v^{k-1} + \sigma K u^k - \sigma w^k}{\sigma}\right\|^2,$$

$$w^{k+1} = \operatorname*{argmin}_w F(w) + \frac{\sigma}{2}\left\| w - \frac{v^k}{\sigma} - 2Ku^{k+1}\right\|^2,$$

$$v^{k+1} = v^k + \sigma(Ku^{k+1} - w^{k+1})$$

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Alternating Direction Method of Multipliers (ADMM)

- Replace variable p^k in all update steps

$$u^{k+1} = \underset{u}{\operatorname{argmin}}\, G(u) + \frac{\sigma}{2}\left\|Ku + \frac{v^{k-1} + \sigma K u^k - \sigma w^k}{\sigma}\right\|^2,$$

$$w^{k+1} = \underset{w}{\operatorname{argmin}}\, F(w) + \frac{\sigma}{2}\left\|w - \frac{v^k}{\sigma} - 2Ku^{k+1}\right\|^2,$$

$$v^{k+1} = v^k + \sigma(Ku^{k+1} - w^{k+1})$$

- Rewrite as:

$$u^{k+1} = \underset{u}{\operatorname{argmin}}\, G(u) + \frac{\sigma}{2}\left\|Ku - w^k + \frac{v^{k-1} + \sigma K u^k}{\sigma}\right\|^2,$$

$$w^{k+1} = \underset{w}{\operatorname{argmin}}\, F(w) + \frac{\sigma}{2}\left\|w - Ku^{k+1} - \frac{v^k + \sigma K u^{k+1}}{\sigma}\right\|^2,$$

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Alternating Direction Method of Multipliers (ADMM)

- Using the following fact we can further rewrite the updates:

$$\operatorname*{argmin}_a \; \frac{\sigma}{2} \left\| a - \frac{b}{\sigma} \right\|^2 = \operatorname*{argmin}_a \; -\langle a, b \rangle + \frac{\sigma}{2} \|a\|^2$$

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$$\underset{a}{\text{argmin}}\ \frac{\sigma}{2}\left\|a - \frac{b}{\sigma}\right\|^2 = \underset{a}{\text{argmin}}\ -\langle a, b\rangle + \frac{\sigma}{2}\|a\|^2$$

- Pulling terms of the squared norm:

$$u^{k+1} = \underset{u}{\text{argmin}}\, G(u) + \langle Ku, v^{k-1} + \sigma K u^k\rangle + \frac{\sigma}{2}\left\|Ku - w^k\right\|^2,$$

$$w^{k+1} = \underset{w}{\text{argmin}}\, F(w) - \langle w, v^k + \sigma K u^{k+1}\rangle + \frac{\sigma}{2}\left\|w - Ku^{k+1}\right\|^2,$$

$$v^{k+1} = v^k + \sigma(Ku^{k+1} - w^{k+1})$$

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- Pulling terms of the squared norm:

$$u^{k+1} = \underset{u}{\operatorname{argmin}}\, G(u) + \langle Ku, v^{k-1} + \sigma K u^k \rangle + \frac{\sigma}{2} \left\| Ku - w^k \right\|^2,$$

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$$u^{k+1} = \underset{u}{\operatorname{argmin}} \, G(u) + \langle Ku, v^{k-1} + \sigma K u^k\rangle + \frac{\sigma}{2}\left\|Ku - w^k\right\|^2,$$

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$$v^{k+1} = v^k + \sigma(Ku^{k+1} - w^{k+1})$$

- Reintroduce $p^{k+1} = v^k + \sigma K u^{k+1}$, can be rewritten as:

$$u^{k+1} = \underset{u}{\operatorname{argmin}} \, G(u) + \langle Ku, p^k\rangle + \frac{\sigma}{2}\left\|Ku - w^k\right\|^2,$$

$$w^{k+1} = \underset{w}{\operatorname{argmin}} \, F(w) - \langle w, p^{k+1}\rangle + \frac{\sigma}{2}\left\|w - Ku^{k+1}\right\|^2,$$

$$p^{k+1} = p^k + \sigma(Ku^{k+1} - w^k)$$

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Alternating Direction Method of Multipliers (ADMM)

- Let $\bar{w}^{k+1} = w^k$:

$$u^{k+1} = \operatorname*{argmin}_u G(u) + \langle Ku, p^k \rangle + \frac{\sigma}{2} \left\| Ku - \bar{w}^{k+1} \right\|^2,$$

$$\bar{w}^{k+2} = \operatorname*{argmin}_w F(w) - \langle w, p^{k+1} \rangle + \frac{\sigma}{2} \left\| w - Ku^{k+1} \right\|^2,$$

$$p^{k+1} = p^k + \sigma(Ku^{k+1} - \bar{w}^{k+1})$$

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$$p^{k+1} = p^k + \sigma(Ku^{k+1} - \bar{w}^{k+1})$$

- Change order of first two iterates:

$$\bar{w}^{k+1} = \operatorname*{argmin}_w F(w) - \langle w, p^k \rangle + \frac{\sigma}{2} \left\| w - Ku^k \right\|^2,$$

$$u^{k+1} = \operatorname*{argmin}_u G(u) + \langle Ku, p^k \rangle + \frac{\sigma}{2} \left\| Ku - \bar{w}^{k+1} \right\|^2,$$

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Alternating Direction Method of Multipliers (ADMM)

- Final update equations:

$$
\begin{aligned}
w^{k+1} &= \operatorname*{argmin}_{w} F(w) - \langle w, p^k \rangle + \frac{\sigma}{2} \left\| w - Ku^k \right\|^2, \\
u^{k+1} &= \operatorname*{argmin}_{u} G(u) + \langle Ku, p^k \rangle + \frac{\sigma}{2} \left\| Ku - w^{k+1} \right\|^2, \\
p^{k+1} &= p^k + \sigma(Ku^{k+1} - w^{k+1})
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[9] Boyd et al., Distributed optimization and statistical learning via the alternating direction method of multipliers, 2011

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- Alternating minimization of the **augmented Lagrangian**:

$$L^{\tau}_{\text{aug}}(u, w, p) = G(u) + F(w) + \langle p, Ku - w \rangle + \frac{\tau}{2} \| Ku - w \|^2$$

[9] Boyd et al., Distributed optimization and statistical learning via the alternating direction method of multipliers, 2011

Alternating Direction Method of Multipliers (ADMM)

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- The method in this form is called Alternating Direction Method of Multipliers (ADMM)

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- The method in this form is called Alternating Direction Method of Multipliers (ADMM)
- It has gained enormous popularity recently [9], over 3458 citations in 5 years

[9] Boyd et al., Distributed optimization and statistical learning via the alternating direction method of multipliers, 2011

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- Splitting methods split problem into simpler subproblems

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Conclusion

- Splitting methods split problem into simpler subproblems
- Many other splitting approaches exist that can explicitly handle differentiable functions (Forward-Backward, Forward-Backward-Forward, Davis-Yin, ...)

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- Splitting methods split problem into simpler subproblems
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- Many relations exist between the primal-dual algorithms, often special cases of one another

Michael Moeller
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Conclusion

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- Many relations exist between the primal-dual algorithms, often special cases of one another
- Depending on the problem structure, better to use either Graph Projection/DRS/ADMM or PDHG (more next week!)

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Conclusion

- Splitting methods split problem into simpler subproblems
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- Many relations exist between the primal-dual algorithms, often special cases of one another
- Depending on the problem structure, better to use either Graph Projection/DRS/ADMM or PDHG (more next week!)
- **Rule of thumb:** Graph Projection/DRS/ADMM few expensive iterations, PDHG many cheap iterations

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