Convex Optimization for Computer Vision
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Weekly Exercises 10

Room 01.09.014
Friday, 8.7.2016, 09:00-11:00
Submission deadline: Wednesday, 6.7.2016, 14:15, Room: 02.09.023

Theory (4 Points)

Exercise 1 (4 points)**.** The Huber penalty $h_\varepsilon : \mathbb{R} \to \mathbb{R}$ is given as

$$h_\varepsilon(x) = \begin{cases} \frac{x^2}{2\varepsilon} & \text{if } |x| \leq \varepsilon, \\ |x| - \frac{\varepsilon}{2} & \text{otherwise.} \end{cases}$$

1. Show that the huber penalty can be expressed as the infimal convolution of the functions $f : \mathbb{R} \to \mathbb{R}$ with $f(x) := \frac{x^2}{2\varepsilon}$ and $g : \mathbb{R} \to \mathbb{R}$ with $g(x) := |x|$:

$$h_\varepsilon(x) = (f \,\square\, g)(x).$$

2. Compute the convex conjugate of the function $t : \mathbb{R}^{2N} \to \mathbb{R}$ defined as

$$t(x) := \sum_{i=1}^{2N} h_\varepsilon(x_i).$$

Solution. 1. For the infimal convolution we have that

$$\left(\frac{(\cdot)^2}{2\varepsilon} \,\square\, |\cdot| \right)(u) = \inf_{v \in \mathbb{R}^n} \frac{1}{2\varepsilon}(u - v)^2 + |v|$$

This is the soft thresholding problem we had on exercise sheet 3. The minimizer for that is attained at

$$v^* = \begin{cases} u + \varepsilon & \text{if } u < -\varepsilon \\ 0 & \text{if } u \in [-\varepsilon, \varepsilon] \\ u - \varepsilon & \text{if } u > \varepsilon. \end{cases}$$

Plugging this in we obtain for the infimum:

$$\left(\frac{(\cdot)^2}{2\varepsilon} \,\square\, |\cdot| \right)(u) = \begin{cases} \frac{1}{2\varepsilon}\varepsilon^2 + |u + \varepsilon| = \frac{\varepsilon}{2} - u - \varepsilon = -\frac{\varepsilon}{2} - u & \text{if } u < -\varepsilon \\ \frac{1}{2\varepsilon}u^2 & \text{if } u \in [-\varepsilon, \varepsilon] \\ \frac{1}{2\varepsilon}\varepsilon^2 + u - \varepsilon = -\frac{\varepsilon}{2} + u & \text{if } u > \varepsilon. \end{cases}$$

which is obviously what we were supposed to show.

2. Since the elements of the sum are independent the sum decouples. That is one can compute the conjugate of the Huber terms seperately. From the notes or the result above we obtain:

$$h_\varepsilon^*(y_i) = \frac{\varepsilon}{2}y_i^2 \; + \; \iota_{|\cdot|\leq 1}(y_i).$$

   The overall conjugate is then given as:

$$h_\varepsilon^*(y) = \frac{\varepsilon}{2}\|y\|_2^2 \; + \; \iota_{|\cdot|_\infty \leq 1}(y).$$

Optimization Challange (12 Points)

Exercise 2 (12 Points)**.** Complete the optimization challange. Only reasonable attempts will be counted.